

CS4102 Recurrence Proofs - Spring 2022

Karatsuba Recurrence: Tree Method

Karatsuba Recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Using the tree method for solving the recurrence, we obtained the sum:

$$\begin{aligned} T(n) &= 8n \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i \\ &= 8n \frac{\left(\frac{3}{2}\right)^{\log_2 n+1} - 1}{\frac{3}{2} - 1} \\ &= 8n \frac{\left(\frac{3}{2}\right)^{\log_2 n+1} - 1}{\frac{1}{2}} \\ &= 16n \left(\left(\frac{3}{2}\right)^{\log_2 n+1} - 1 \right) \\ &= 16n(2^{\log_2 3-1})^{\log_2 n+1} - 16n \\ &= 16n(2^{\log_2 3 \cdot \log_2 n - \log_2 n + \log_2 3 - 1}) - 16n \\ &= 16n((2^{\log_2 n})^{\log_2 3} \cdot 2^{-\log_2 n} \cdot 2^{\log_2 3} \cdot 2^{-1}) - 16n \\ &= 16n \left(n^{\log_2 3} \cdot \frac{1}{n} \cdot 3 \cdot \frac{1}{2} \right) - 16n \\ &= 24n^{\log_2 3} - 16n \\ &= \Theta(n^{\log_2 3}) \\ &\approx \Theta(n^{1.585}) \end{aligned}$$

Karatsuba, Guess and Check, Loose Bound

Karatsuba Recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Goal:

$$T(n) \leq 3000n^{1.6}$$

Base Case:

$$T(1) = 8 \leq 3000$$

Hypothesis:

$$\forall n < x_0, T(n) \leq 3000n^{1.6}$$

Inductive Step:

$$\begin{aligned} T(x_0 + 1) &= 3T\left(\frac{x_0 + 1}{2}\right) + 8(x_0 + 1) \\ &\leq 3 \left(3000 \left(\frac{x_0 + 1}{2}\right)^{1.6} \right) + 8(x_0 + 1) \\ &= \frac{3}{2^{1.6}} \cdot 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) \\ &\leq 0.997 \cdot 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) \\ &= (1 - 0.003) \cdot 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) \\ &= 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) - 0.003 \cdot 3000(x_0 + 1)^{1.6} \\ &= 3000(x_0 + 1)^{1.6} + 8(x_0 + 1) - 9(x_0 + 1)^{1.6} \\ &\leq 3000(x_0 + 1)^{1.6} \end{aligned}$$

Karatsuba, Guess and Check, Tight Bound

Karatsuba Recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Goal:

$$T(n) \leq 24n^{\log_2 3} - 16n$$

Base Case: by inspection

Hypothesis:

$$\forall n < x_0, T(n) \leq 24n^{\log_2 3} - 16n$$

Inductive Step:

$$\begin{aligned} T(x_0 + 1) &= 3T\left(\frac{x_0 + 1}{2}\right) + 8(x_0 + 1) \\ &\leq 3\left(24\left(\frac{x_0 + 1}{2}\right)^{\log_2 3} - 16\frac{x_0 + 1}{2}\right) + 8(x_0 + 1) \\ &= 3\left(\frac{24}{3}(x_0 + 1)^{\log_2 3} - 8(x_0 + 1)\right) + 8(x_0 + 1) \\ &= 24(x_0 + 1)^{\log_2 3} - 24(x_0 + 1) + 8(x_0 + 1) \\ &= 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1) \end{aligned}$$

Bad MergeSort, Guess and Check

MergeSort Recurrence:

$$T(n) = 2T\left(\frac{n}{2}\right) + 209n$$

Goal:

$$T(n) \leq 209n \log_2 n$$

Base Case: by inspection

Hypothesis:

$$\forall n < x_0, T(n) \leq 209n \log_2 n$$

Inductive Step:

$$\begin{aligned} T(x_0 + 1) &= 2T\left(\frac{x_0 + 1}{2}\right) + 209(x_0 + 1) \\ &\leq 2\left(209 \frac{x_0 + 1}{2} \log_2 \frac{x_0 + 1}{2}\right) + 209(x_0 + 1) \\ &= 209(x_0 + 1) \log_2 \frac{x_0 + 1}{2} + 209(x_0 + 1) \\ &= 209(x_0 + 1) \left(\log_2(x_0 + 1) + \log_2 \frac{1}{2}\right) + 209(x_0 + 1) \\ &= 209(x_0 + 1) (\log_2(x_0 + 1) - 1) + 209(x_0 + 1) \\ &= 209(x_0 + 1) \log_2(x_0 + 1) - 209(x_0 + 1) + 209(x_0 + 1) \\ &= 209(x_0 + 1) \log_2(x_0 + 1) \end{aligned}$$

MergeSort, Guess and Check

MergeSort Recurrence:

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Goal:

$$T(n) \leq n \log_2 n$$

Base Case: by inspection

Hypothesis:

$$\forall n < x_0, T(n) \leq n \log_2 n$$

Inductive Step:

$$\begin{aligned} T(x_0 + 1) &= 2T\left(\frac{x_0 + 1}{2}\right) + (x_0 + 1) \\ &\leq 2\left(\frac{x_0 + 1}{2} \log_2 \frac{x_0 + 1}{2}\right) + x_0 + 1 \\ &= (x_0 + 1) \log_2 \frac{x_0 + 1}{2} + x_0 + 1 \\ &= (x_0 + 1)(\log_2(x_0 + 1) + \log_2 \frac{1}{2}) + x_0 + 1 \\ &= (x_0 + 1)(\log_2(x_0 + 1) - 1) + x_0 + 1 \\ &= (x_0 + 1) \log_2(x_0 + 1) - (x_0 + 1) + x_0 + 1 \\ &= (x_0 + 1) \log_2(x_0 + 1) \end{aligned}$$

Master Theorem Case 1

Recurrence:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Assumption:

$$f(n) = O(n^{\log_b a - \varepsilon}) \Rightarrow f(n) \leq c \cdot n^{\log_b a - \varepsilon}$$

To Show:

$$T(n) = O(n^{\log_b a})$$

Proof: (let $L = \log_b n$, i.e. the height of the recurrence tree)

$$\begin{aligned} T(n) &= f(n) + af\left(\frac{n}{b}\right) + a^2f\left(\frac{n}{b^2}\right) + \dots + a^L f\left(\frac{n}{b^L}\right) \\ &\leq c((n)^{\log_b a - \varepsilon} + a\left(\frac{n}{b}\right)^{\log_b a - \varepsilon} + a^2\left(\frac{n}{b^2}\right)^{\log_b a - \varepsilon} + \dots + a^{L-1}\left(\frac{n}{b^{L-1}}\right)^{\log_b a - \varepsilon}) + a^L f(1) \\ &= cn^{\log_b a - \varepsilon} \left(1 + \frac{a}{b^{\log_b a - \varepsilon}} + \frac{a^2}{b^{2\log_b a - \varepsilon}} + \dots + \frac{a^{L-1}}{b^{(L-1)\log_b a - \varepsilon}}\right) + a^L f(1) \\ &= cn^{\log_b a - \varepsilon} (1 + b^\varepsilon + b^{2\varepsilon} + \dots + b^{(L-1)\varepsilon}) + a^L f(1) \\ &= cn^{\log_b a - \varepsilon} \left(\frac{b^{\varepsilon L} - 1}{b^\varepsilon - 1}\right) + a^L f(1) \\ &= cn^{\log_b a - \varepsilon} \left(\frac{b^{\varepsilon \log_b n} - 1}{b^\varepsilon - 1}\right) + a^L f(1) \\ &= cn^{\log_b a - \varepsilon} \left((n^\varepsilon - 1) \cdot \frac{1}{b^\varepsilon - 1}\right) + a^{\log_b n} f(1) \\ &= cn^{\log_b a - \varepsilon} ((n^\varepsilon - 1) \cdot c_2) + n^{\log_b a} \cdot c_3 \\ &= c_4 n^{\log_b a} - c_4 n^{\log_b a - \varepsilon} + n^{\log_b a} \cdot c_3 \\ &= O(n^{\log_b a}) \end{aligned}$$