CS4102 Algorithms

Spring 2022

Warm up

Show that finding the minimum of an unordered list requires $\Omega(n)$ comparisons

Find Min, Lower Bound Proof

Show that finding the minimum of an unordered list requires $\Omega(n)$ comparisons

Suppose (toward contradiction) that there is an algorithm for Find Min that does fewer than $\frac{n}{2} = \Omega(n)$ comparisons.

This means there is at least one "uncompared" element
We can't know that this element wasn't the min!

2	8	19	20	[redacted]	3	9	-4
0	1	2	3	4	5	6	7

Announcements

- Homework schedule on course website
  - Unit A Basic HW2 now available
  - Unit A Advanced and Programming HW now available
  - Unit A Programming submission opens Wednesday
- TA Office Hours
  - 7-10pm Sun-Thurs in Ols 011
  - Online hours also available
- Unit A Exam: Tuesday, February 22, in class

Today's Keywords

- Sorting
- Linear time Sorting
- Counting Sort
- Radix Sort
- Maximum Sum Continuous Subarray

Sorting, so far

- Sorting algorithms we have discussed:
  - Mergesort $O(n \log n)$
  - Quicksort $O(n \log n)$
- Other sorting algorithms (will discuss):
  - Bubblesort $O(n^2)$
  - Insertionsort $O(n^2)$
  - Heapsort $O(n \log n)$

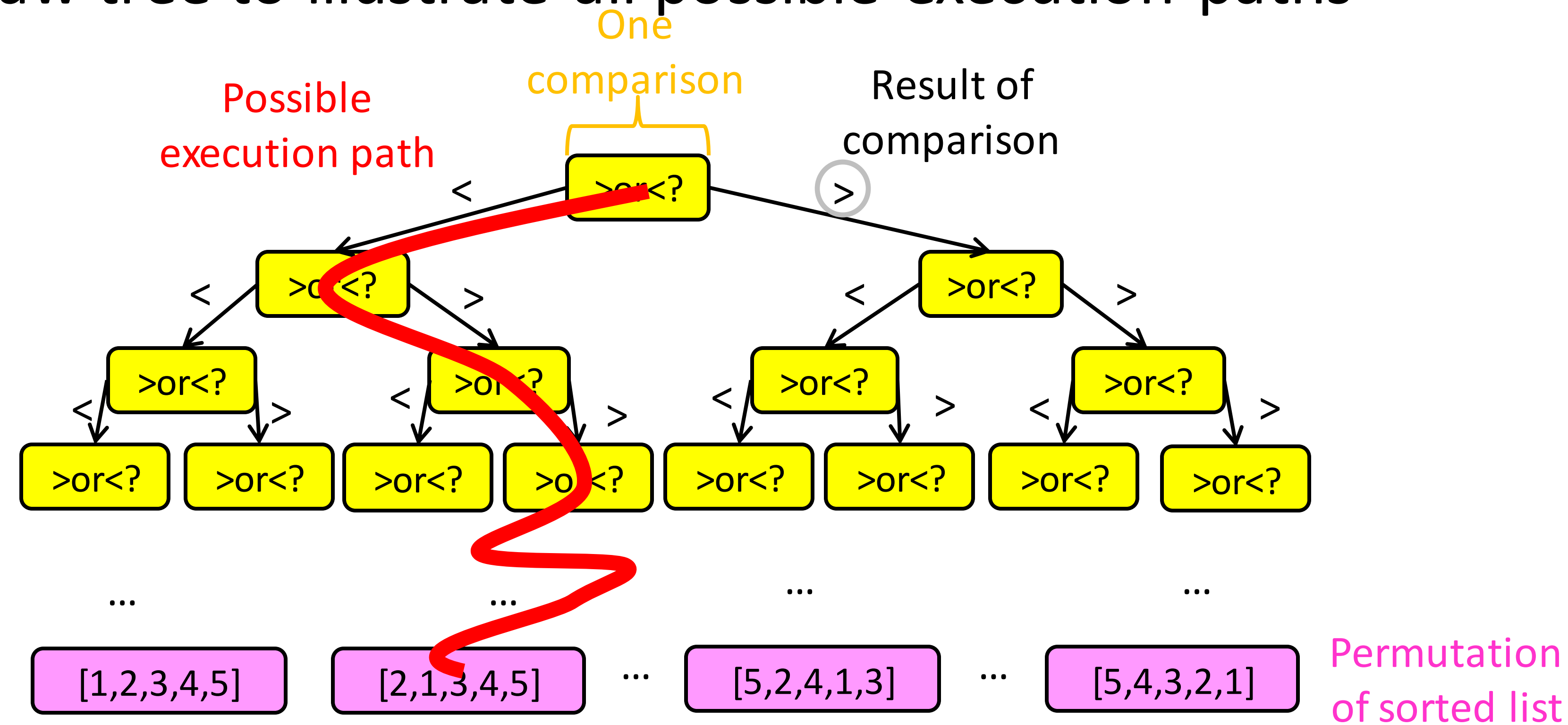
Can we do better than $O(n \log n)$?

Worst Case Lower Bounds

- Prove that there is no algorithm which can sort faster than $O(n \log n)$
- Non-existence proof!
  - Very hard to do

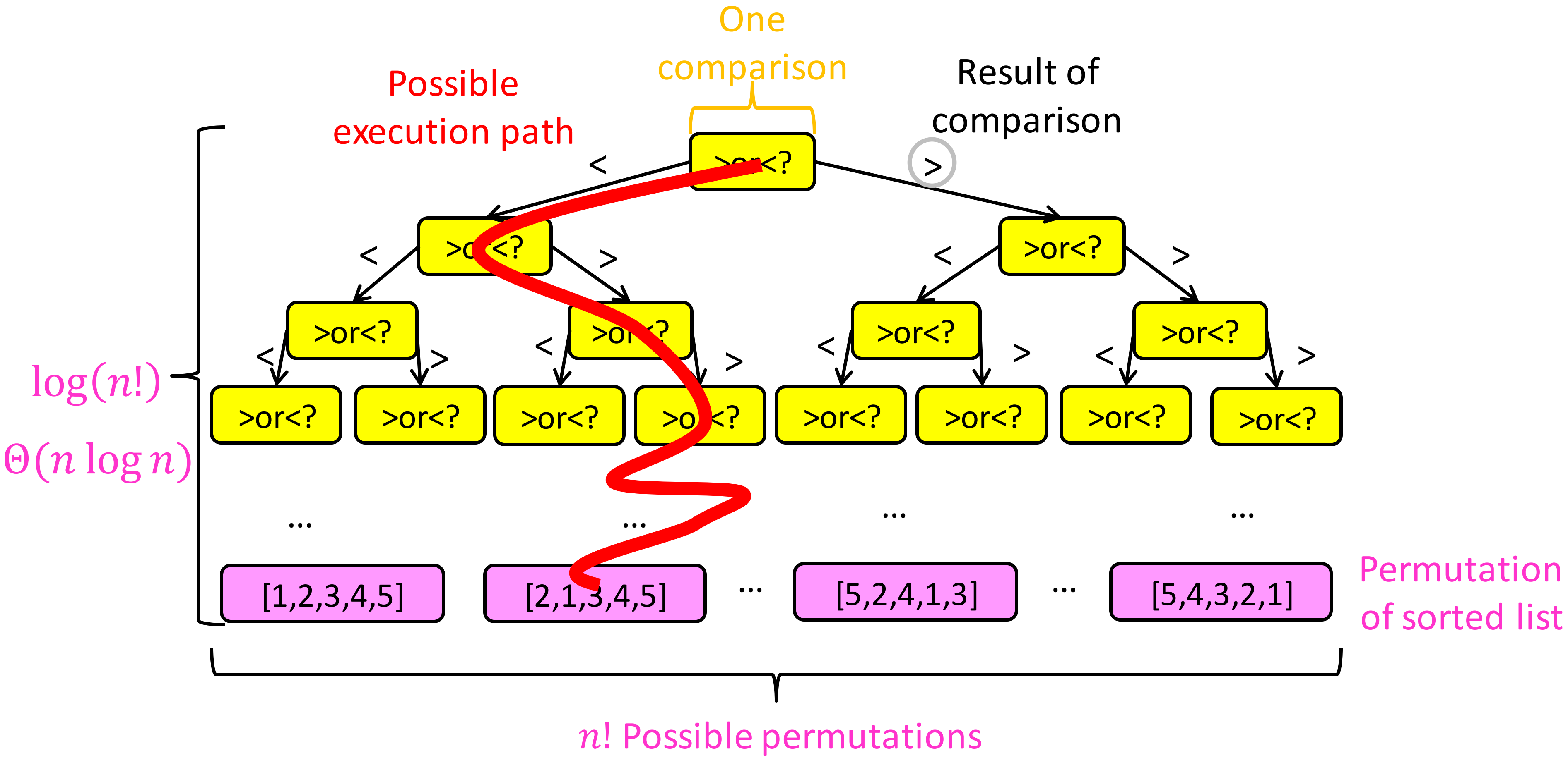
Strategy: Decision Tree

- Sorting algorithms use comparisons to figure out the order of input elements
- Draw tree to illustrate all possible execution paths



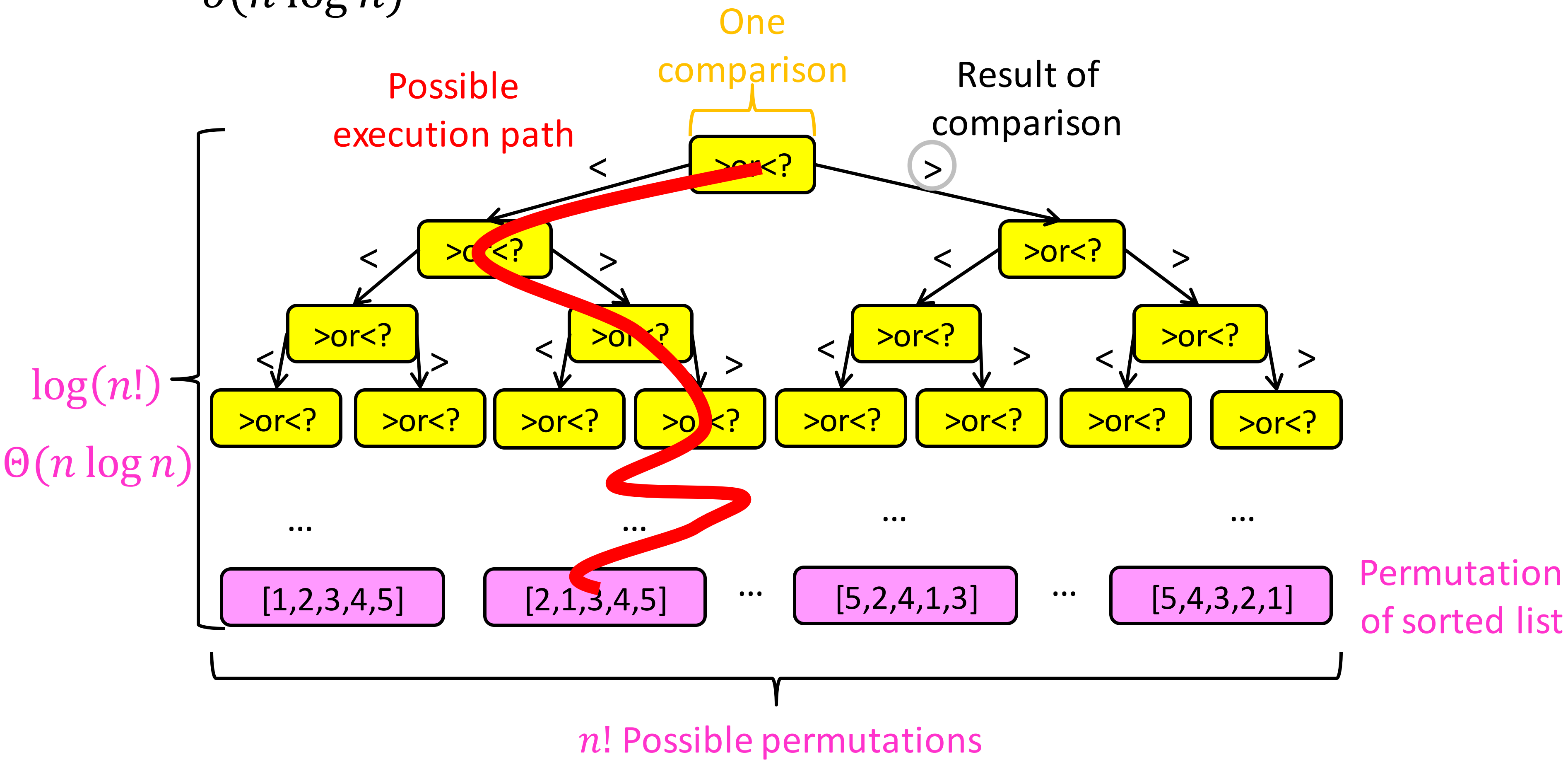
Strategy: Decision Tree

- Worst case run time is the longest execution path
- i.e., “height” of the decision tree



Strategy: Decision Tree

- Conclusion: Worst Case Optimal run time of sorting is $\Theta(n \log n)$
  - There is no (comparison-based) sorting algorithm with run time $o(n \log n)$



Sorting, so far

- Sorting algorithms we have discussed:
  - Mergesort $O(n \log n)$ Optimal!
  - Quicksort $O(n \log n)$ Optimal!
- Other sorting algorithms (will discuss):
  - Bubblesort $O(n^2)$
  - Insertionsort $O(n^2)$
  - Heapsort $O(n \log n)$ Optimal!

Speed Isn't Everything

Important properties of sorting algorithms:

- Run Time
  - Asymptotic Complexity
  - Constants
- In Place (or In-Situ)
  - Done with only constant additional space
- Adaptive
  - Faster if list is nearly sorted
- Stable
  - Equal elements remain in original order
- Parallelizable
  - Runs faster with multiple computers

Mergesort

- **Divide**:
  - Break n-element list into two lists of $^{n}/_{2}$ elements
- **Conquer**:
  - If $n > 1$: Sort each sublist recursively
  - If $n = 1$: List is already sorted (base case)
- **Combine**:
  - Merge together sorted sublists into one sorted list

Run Time?

$\Theta(n \log n)$

Optimal!

In Place?	Adaptive?	Stable?
No	No	Yes! (usually)

Merge

- **Combine**: Merge sorted sublists into one sorted list
- We have:
  - 2 sorted lists (L_1, L_2)
  - 1 output list (L_{out})

While (L_1 and L_2 not empty):

  If $L_1[0] \leq L_2[0]$:

    L_{out}.append(L_1.pop())

  Else:

    L_{out}.append(L_2.pop())

L_{out}.append(L_1)

L_{out}.append(L_2)

Stable:
If elements are equal, leftmost comes first

Mergesort

- **Divide**:
  - Break n-element list into two lists of $^n/_2$ elements
- **Conquer**:
  - If $n > 1$: Sort each sublist recursively
  - If $n = 1$: List is already sorted (base case)
- **Combine**:
  - Merge together sorted sublists into one sorted list

Run Time?

$\Theta(n \log n)$

Optimal!

In Place?	Adaptive?	Stable?	Parallelizable?
No	No	Yes! (usually)	Yes!

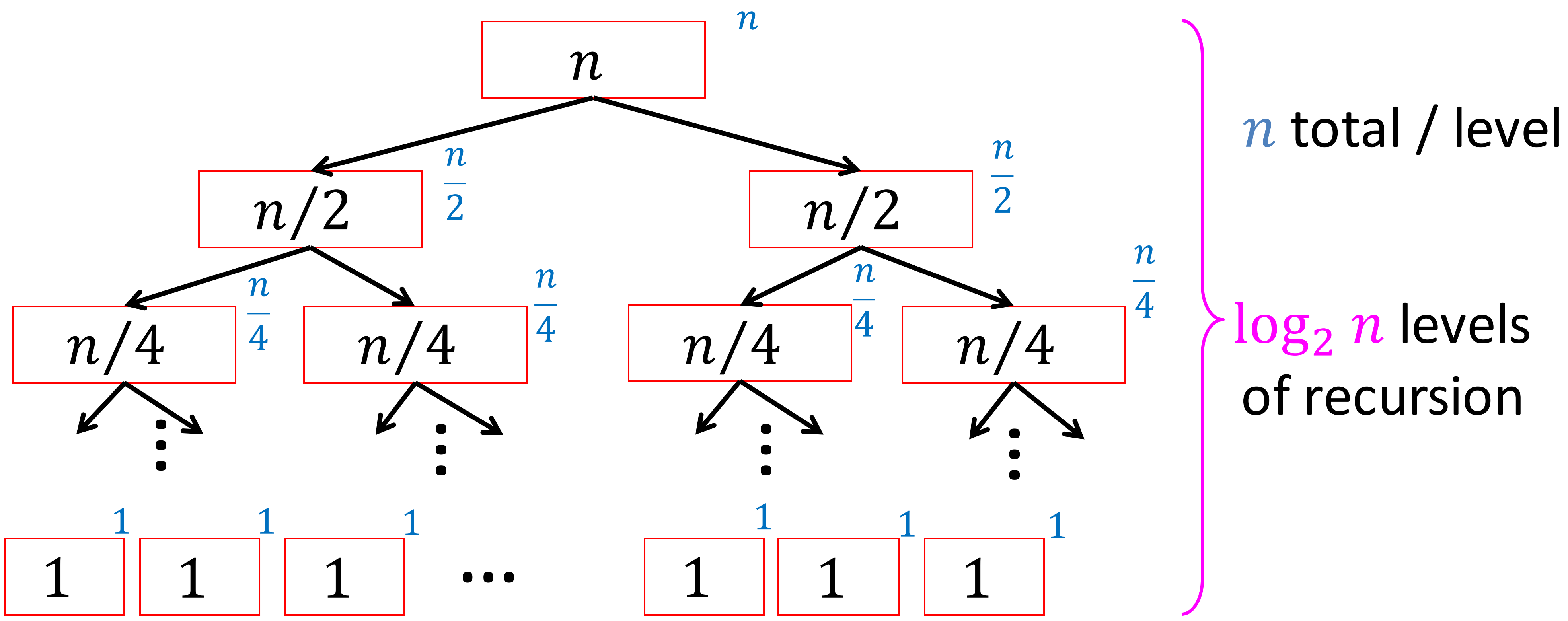
Mergesort

- **Divide**:
  - Break n-element list into two lists of $n/2$ elements
- **Conquer**:
  - If $n > 1$:
    - Sort each sublist recursively
  - If $n = 1$:
    - List is already sorted (base case)
- **Combine**:
  - Merge together sorted sublists into one sorted list

Parallelizable: Allow different machines to work on each sublist

Mergesort (Sequential)

$$T(n) = 2T(\frac{n}{2}) + n$$

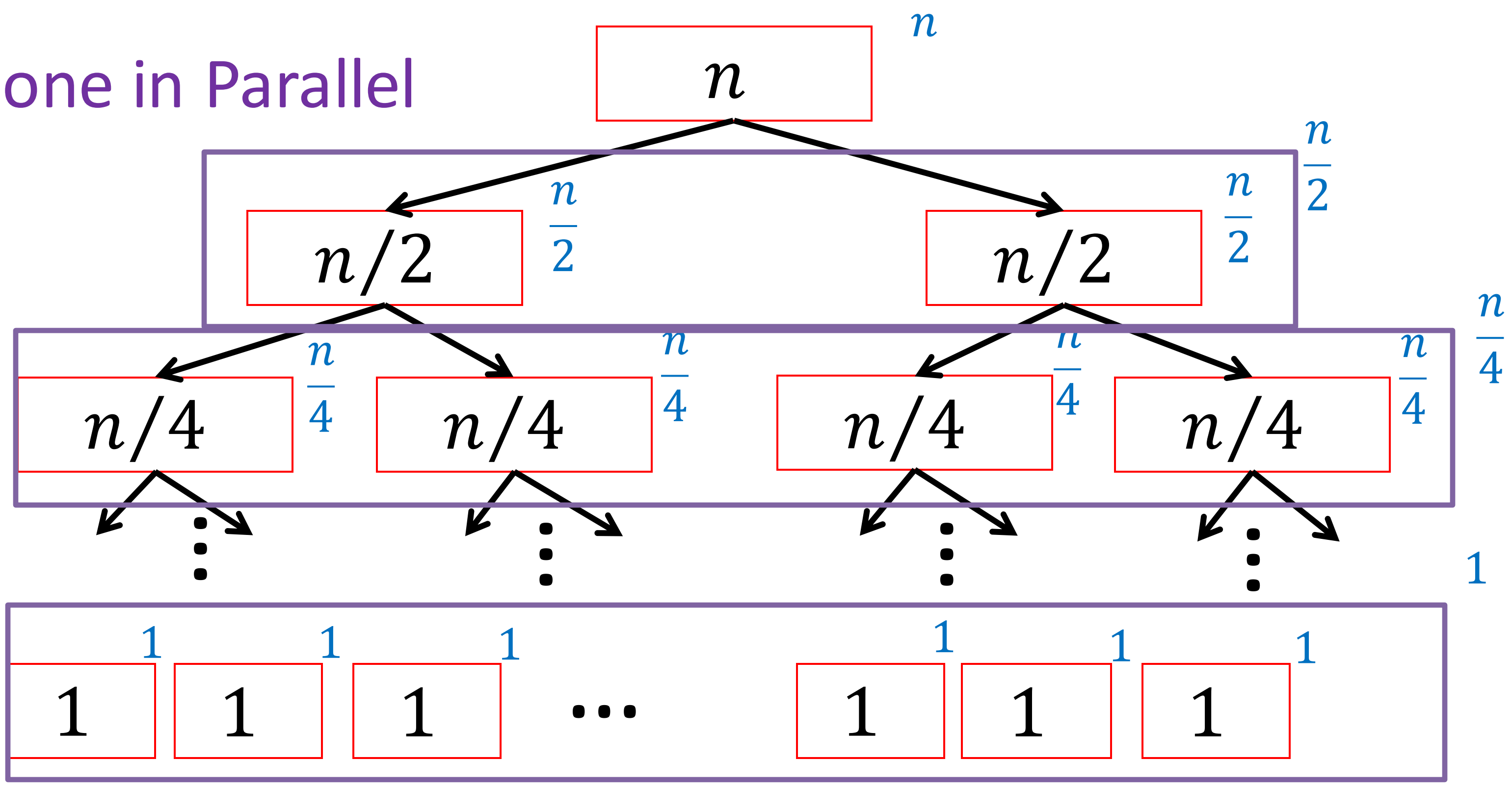


Run Time: $\Theta(n \log n)$

Mergesort (Parallel)

$$T(n) = T(\frac{n}{2}) + n$$

Done in Parallel



Run Time: $\Theta(n)$

Quicksort

Idea: pick a partition element, recursively sort two sublists around that element

- Divide: select an element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

Run Time?

$\Theta(n \log n)$

(almost always)

Better constants than Mergesort

In Place?	Adaptive?	Stable?	Parallelizable?
kinda	No!	No	Yes!
Uses stack for recursive calls			

Bubble Sort

Idea: March through list, swapping adjacent elements if out of order, repeat until sorted

8	**5**	7	9	12	10	1	2	4	3	6	11
5	**8**	**7**	9	12	10	1	2	4	3	6	11
5	7	**8**	**9**	12	10	1	2	4	3	6	11
5	7	8	**9**	**12**	10	1	2	4	3	6	11

Bubble Sort

- Idea: March through list, swapping adjacent elements if out of order, repeat until sorted

Run Time?

$\Theta(n^2)$

Constants worse than Insertion Sort

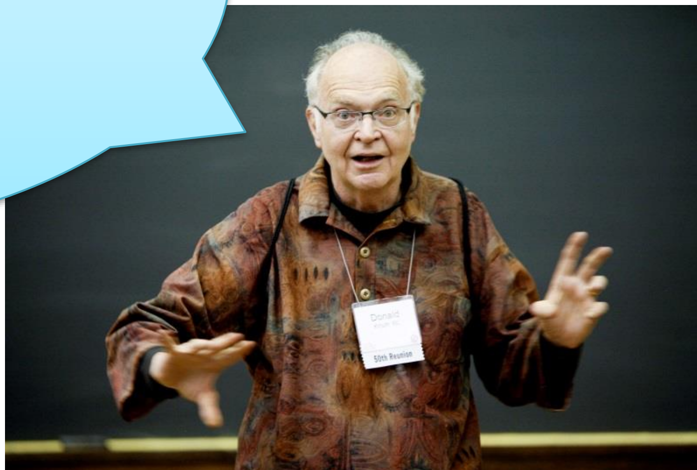
In Place?

Yes

Adaptive?

Kinda

"Compared to straight insertion […], bubble sorting requires a more complicated program and takes about twice as long!" –Donald Knuth



Bubble Sort is “almost” Adaptive

Idea: March through list, swapping adjacent elements if out of order

1	2	3	4	5	6	7	8	9	10	11	12

1	2	3	4	5	6	7	8	9	10	11	12

Only makes one “pass”

2	3	4	5	6	7	8	9	10	11	12	1

After one “pass”

2	3	4	5	6	7	8	9	10	11	1	12

Requires n passes, thus is $O(n^2)$

Bubble Sort

- Idea: March through list, swapping adjacent elements if out of order, repeat until sorted

Run Time?

$\Theta(n^2)$

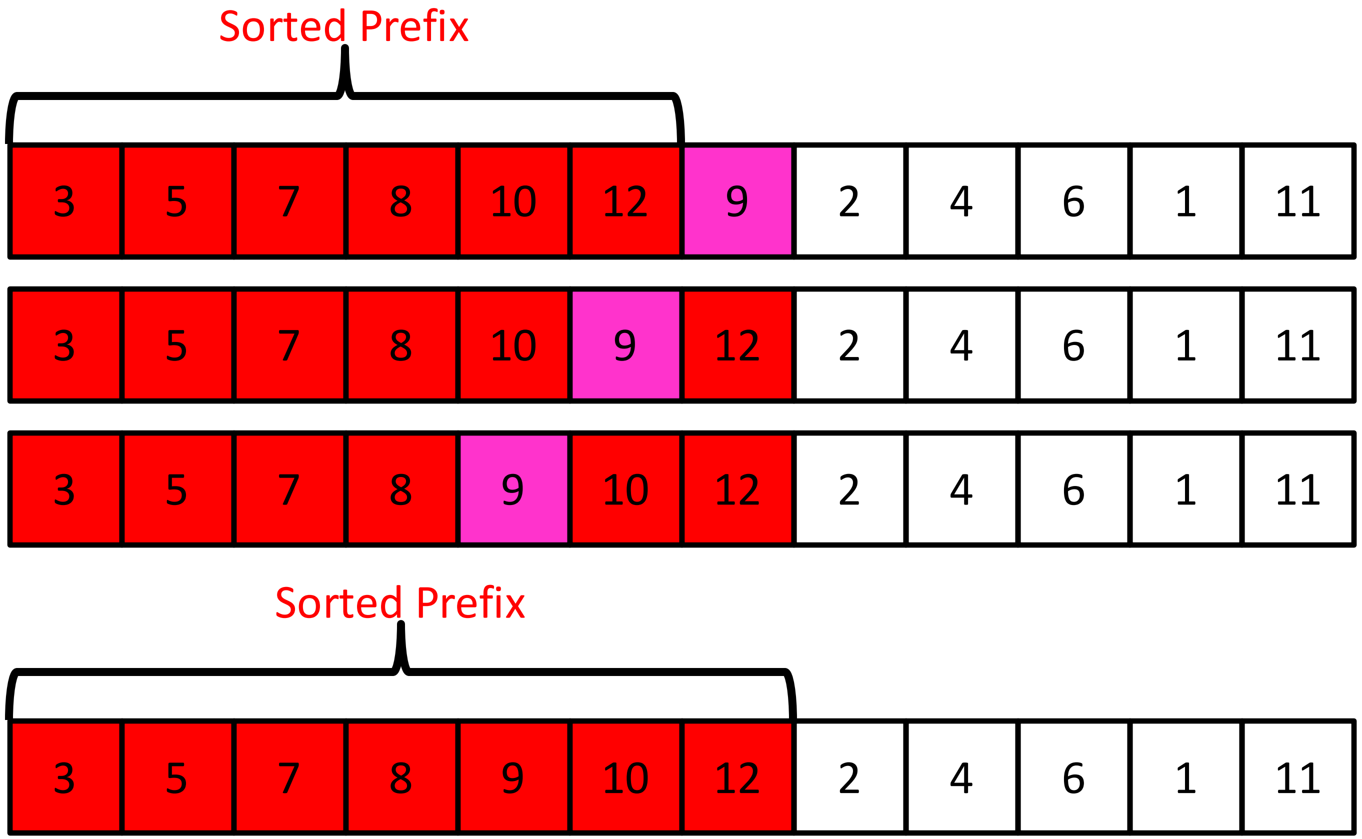
Constants worse than Insertion Sort

In Place?	Adaptive?	Stable?	Parallelizable?
Yes!	~~Kinda~~ Not really	Yes	No

"the bubble sort seems to have nothing to recommend it, except a catchy name and the fact that it leads to some interesting theoretical problems" –Donald Knuth, The Art of Computer Programming

Insertion Sort

Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element



Insertion Sort

- Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element

Run Time?

$\Theta(n^2)$

(but with very small constants)

Great for short lists!

In Place?

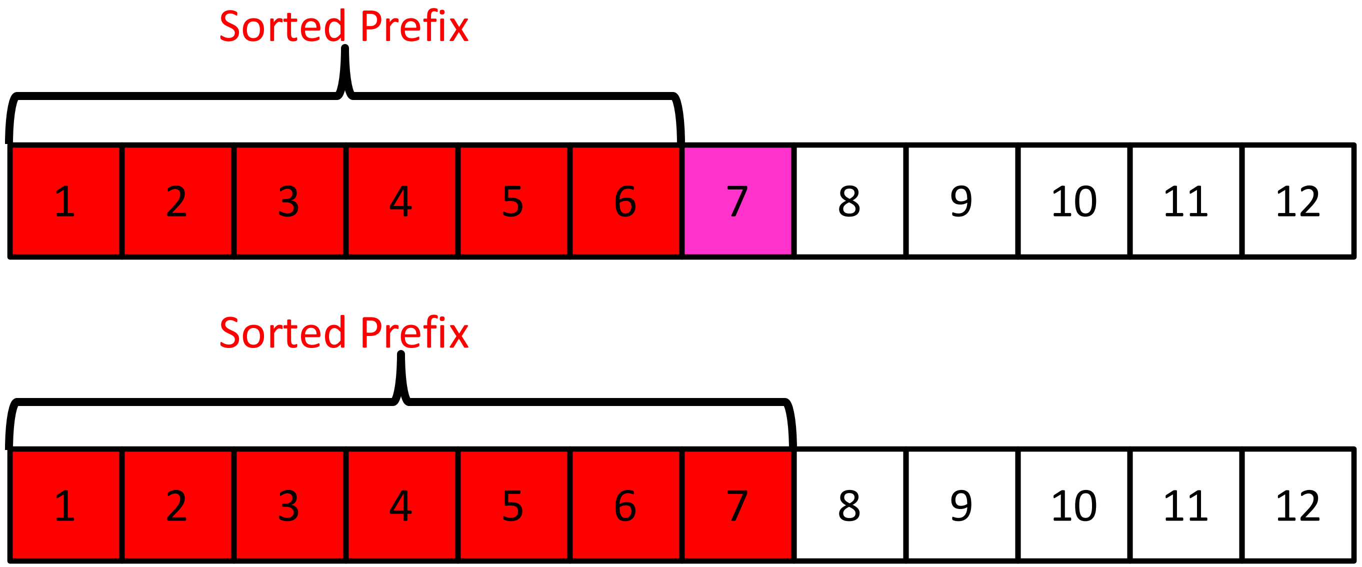
Yes!

Adaptive?

Yes

Insertion Sort is Adaptive

Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element



Only one comparison needed per element!   Runtime: $O(n)$

Insertion Sort

- Idea: Maintain a sorted list prefix, extend that prefix by “inserting” the next element

Run Time?

$\Theta(n^2)$

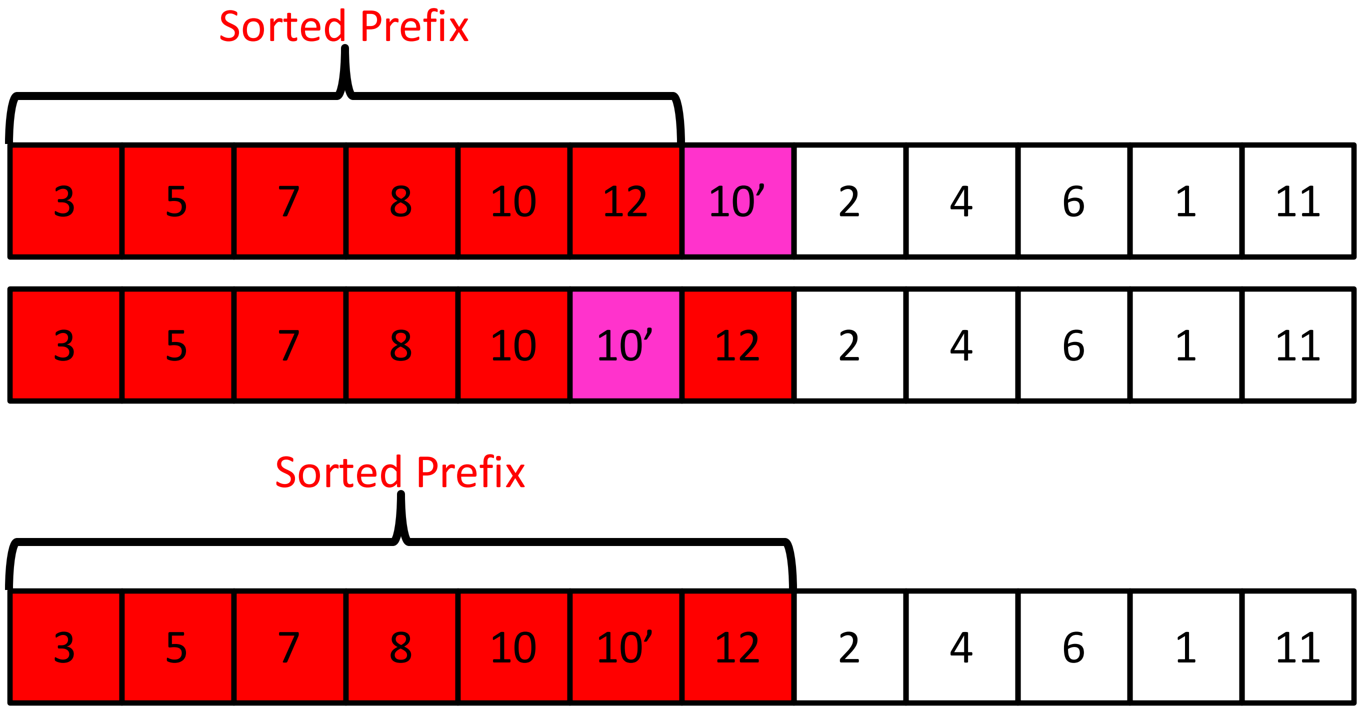
(but with very small constants)

Great for short lists!

In Place?	Adaptive?	Stable?
Yes!	Yes	Yes

Insertion Sort is Stable

- Idea: Maintain a sorted list prefix, extend that prefix by “inserting” the next element



The “second” 10 will stay to the right

Insertion Sort

- Idea: Maintain a sorted list prefix, extend that prefix by “inserting” the next element

Run Time?

$\Theta(n^2)$

(but with very small constants)

Great for short lists!

In Place?	Adaptive?	Stable?	Parallelizable?
Yes!	Yes	Yes	No

Can sort a list as it is received, i.e., don’t need the entire list to begin sorting

Online?

Yes

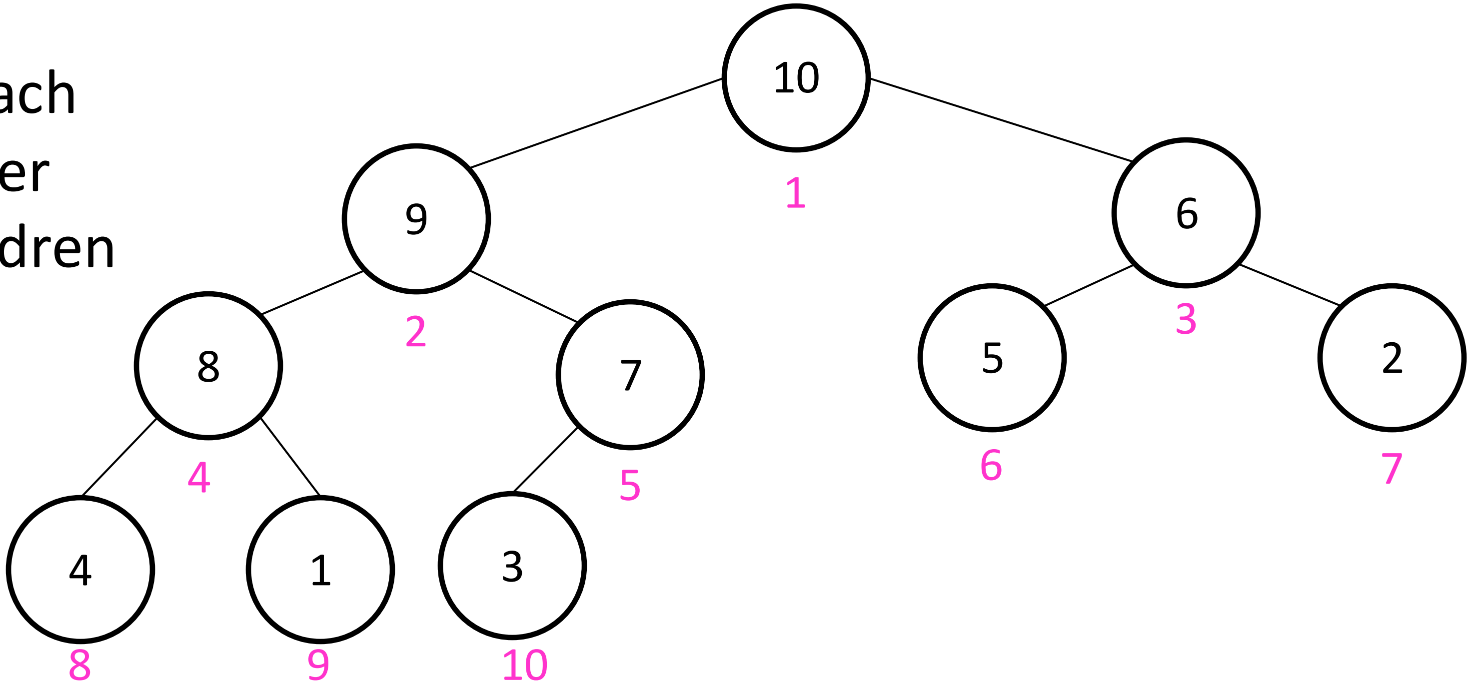
“All things considered, it’s actually a pretty good sorting algorithm!” –Nate Brunelle

Heap Sort

- Idea: Build a Heap, repeatedly extract max element from the heap to build sorted list Right-to-Left

0	1	2	3	4	5	6	7	8	9	10
	10	9	6	8	7	5	2	4	1	3

Max Heap Property: Each node is larger than its children

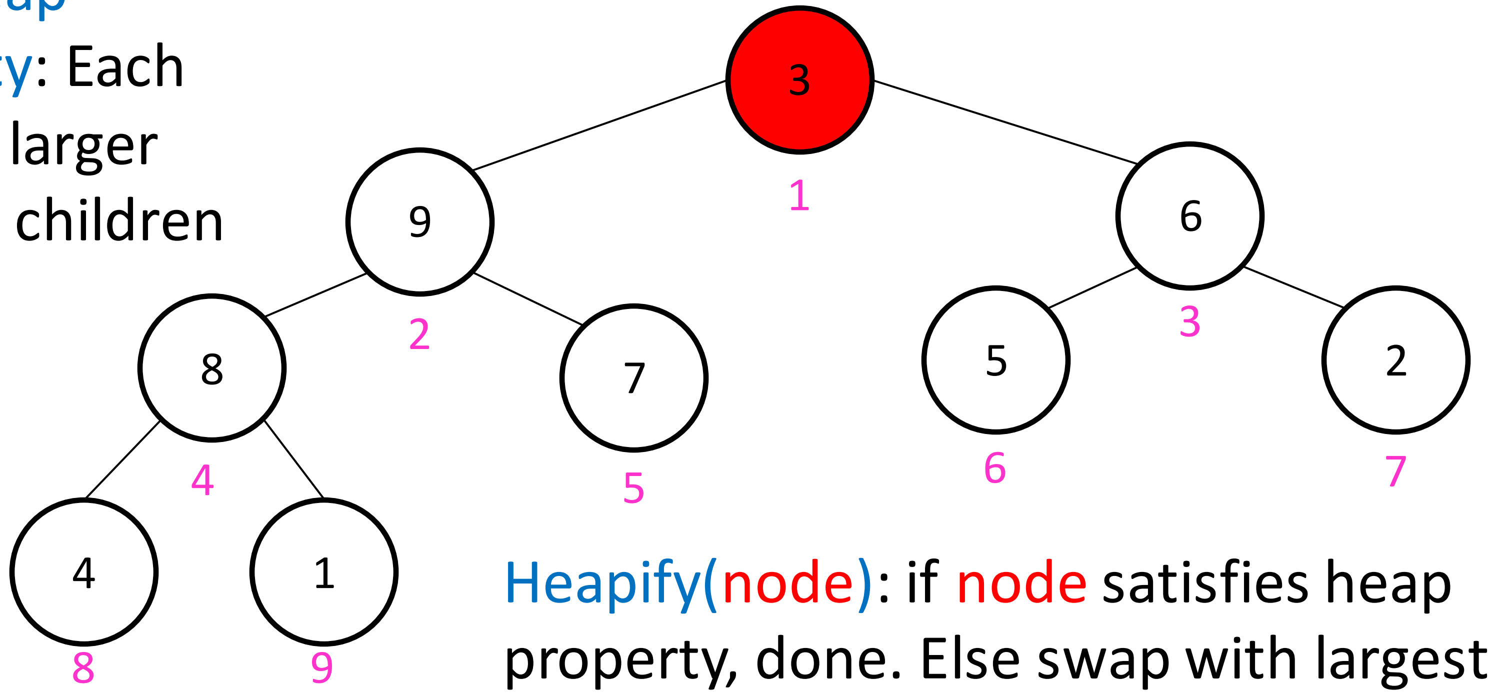


Heap Sort

- Remove the Max element (i.e. the root) from the Heap: replace with last element, call Heapify(root)

	3	9	6	8	7	5	2	4	1	
0	1	2	3	4	5	6	7	8	9	10

Max Heap Property: Each node is larger than its children



Heapify(node): if node satisfies heap property, done. Else swap with largest child and recurse on that subtree

Heap Sort

- Remove the Max element (i.e. the root) from the Heap: replace with last element, call Heapify(root)

	9	3	6	8	7	5	2	4	1	
0	1	2	3	4	5	6	7	8	9	10

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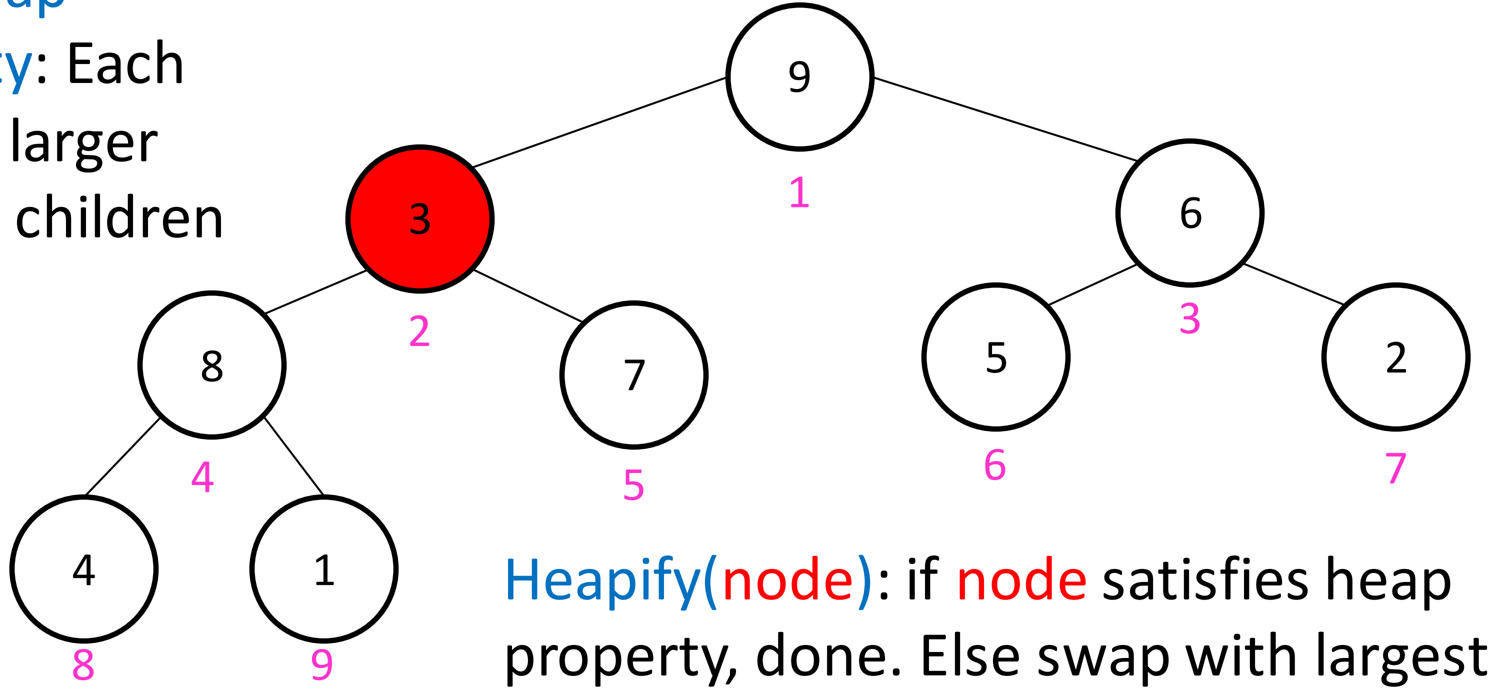

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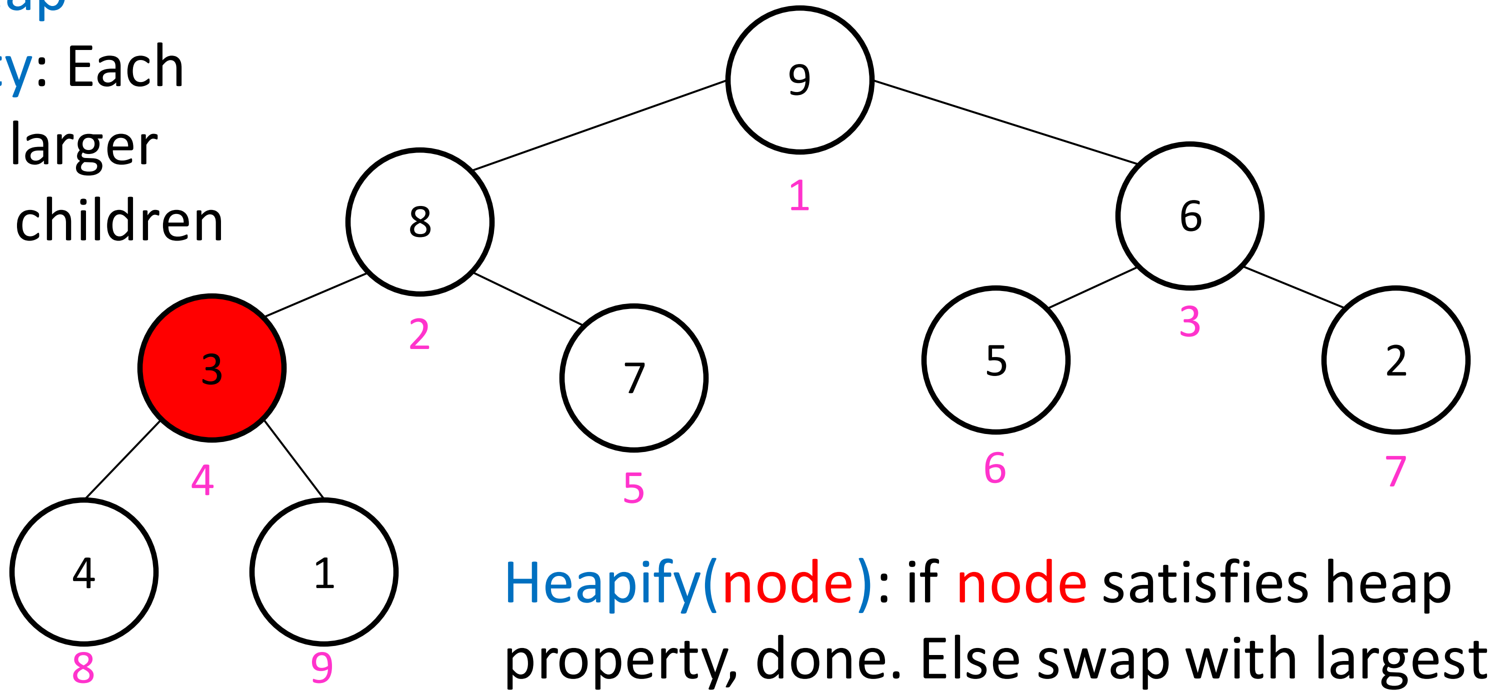

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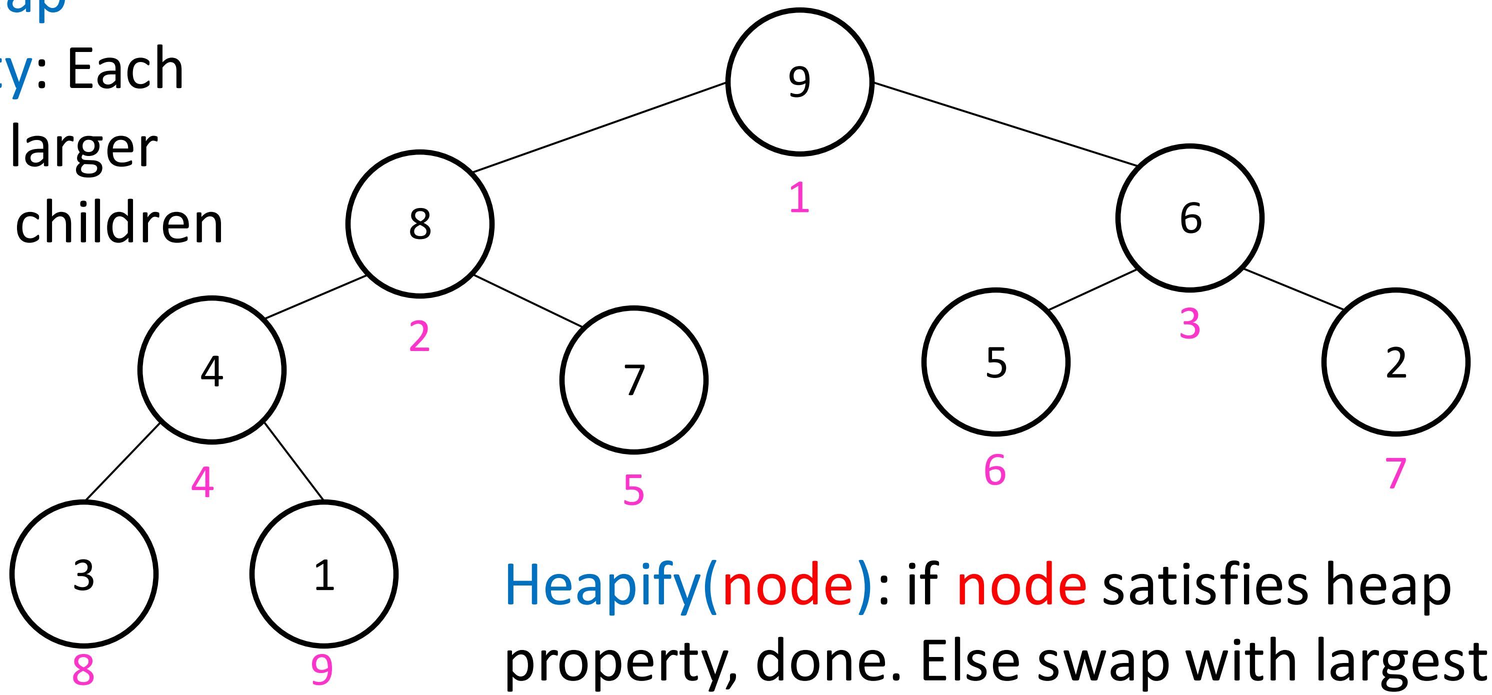

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Run Time?

$\Theta(n \log n)$

Constants worse than Quick Sort

In Place?

Yes!

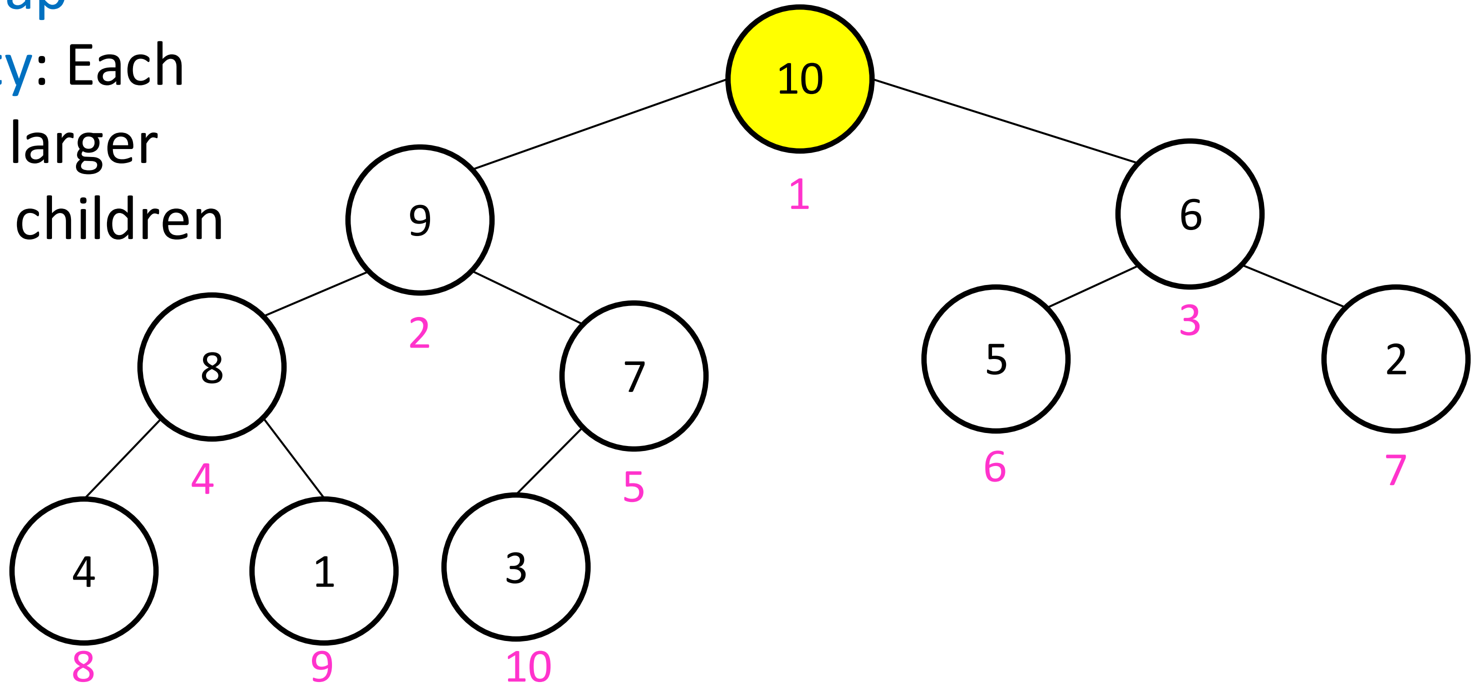
When removing an element from the heap, move it to the (now unoccupied) end of the list

In Place Heap Sort

- Idea: When removing an element from the heap, move it to the (now unoccupied) end of the list

	10	9	6	8	7	5	2	4	1	3
0	1	2	3	4	5	6	7	8	9	10

Max Heap Property: Each node is larger than its children



In Place Heap Sort

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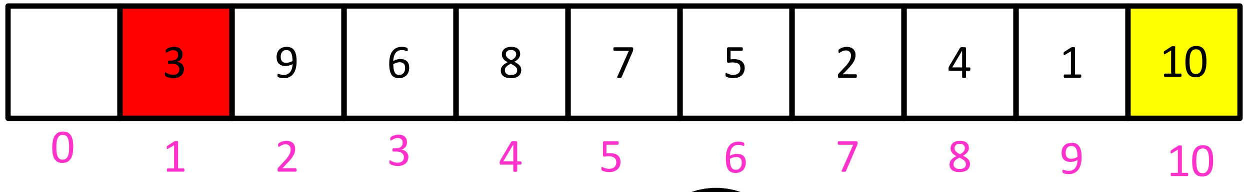

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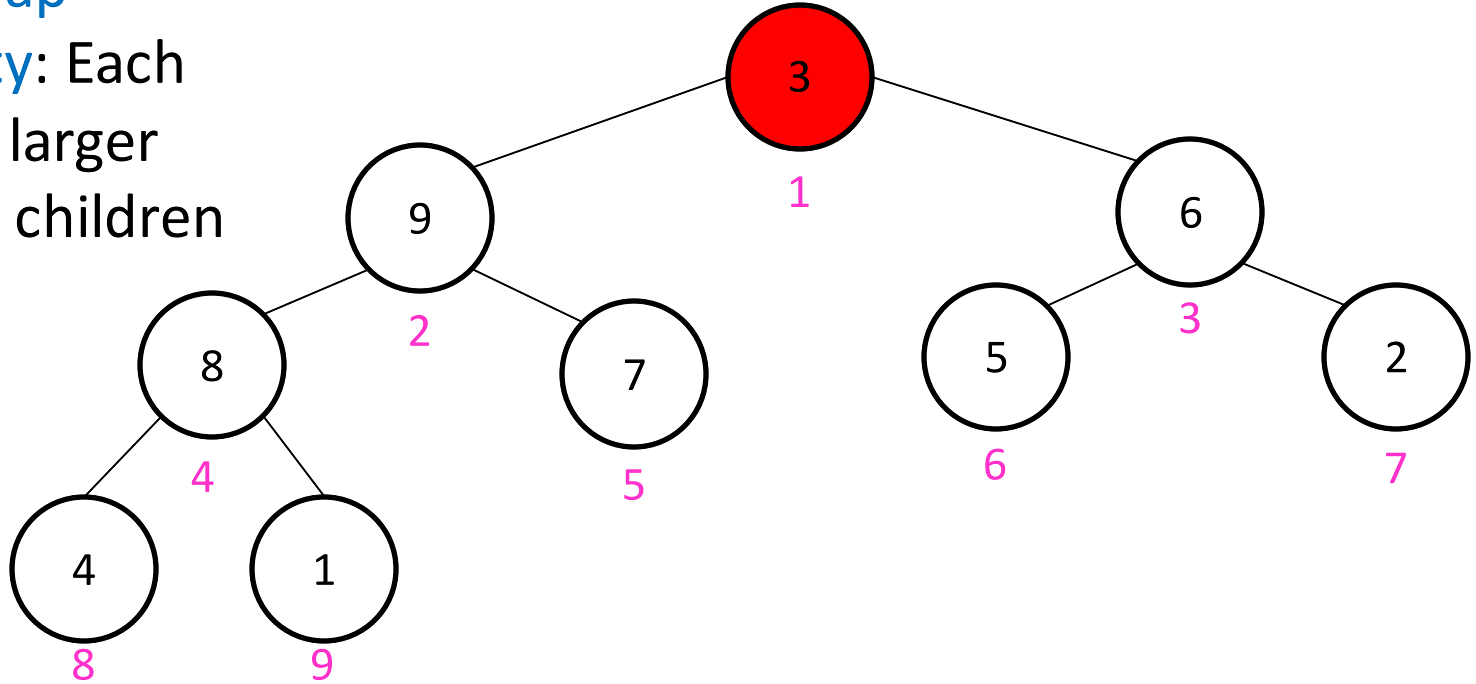

In Place Heap Sort

- Idea: When removing an element from the heap, move it to the (now unoccupied) end of the list

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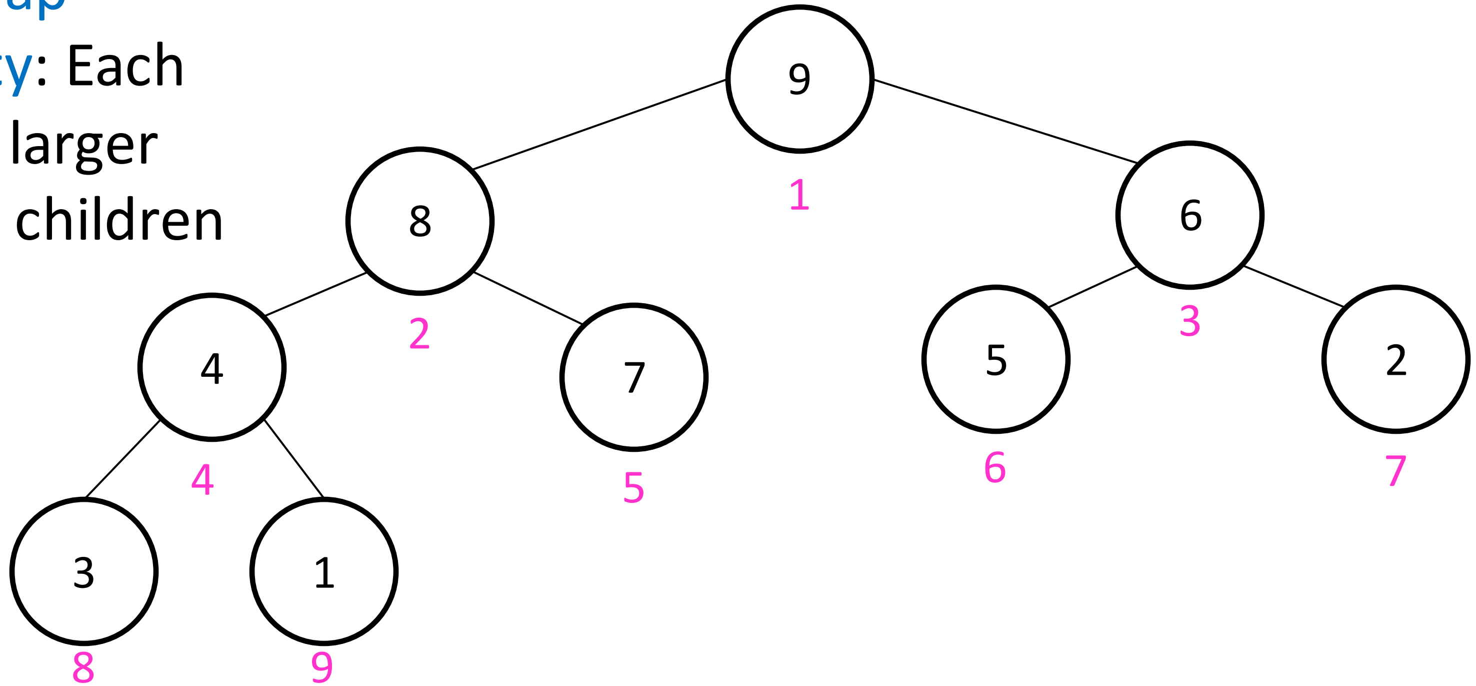
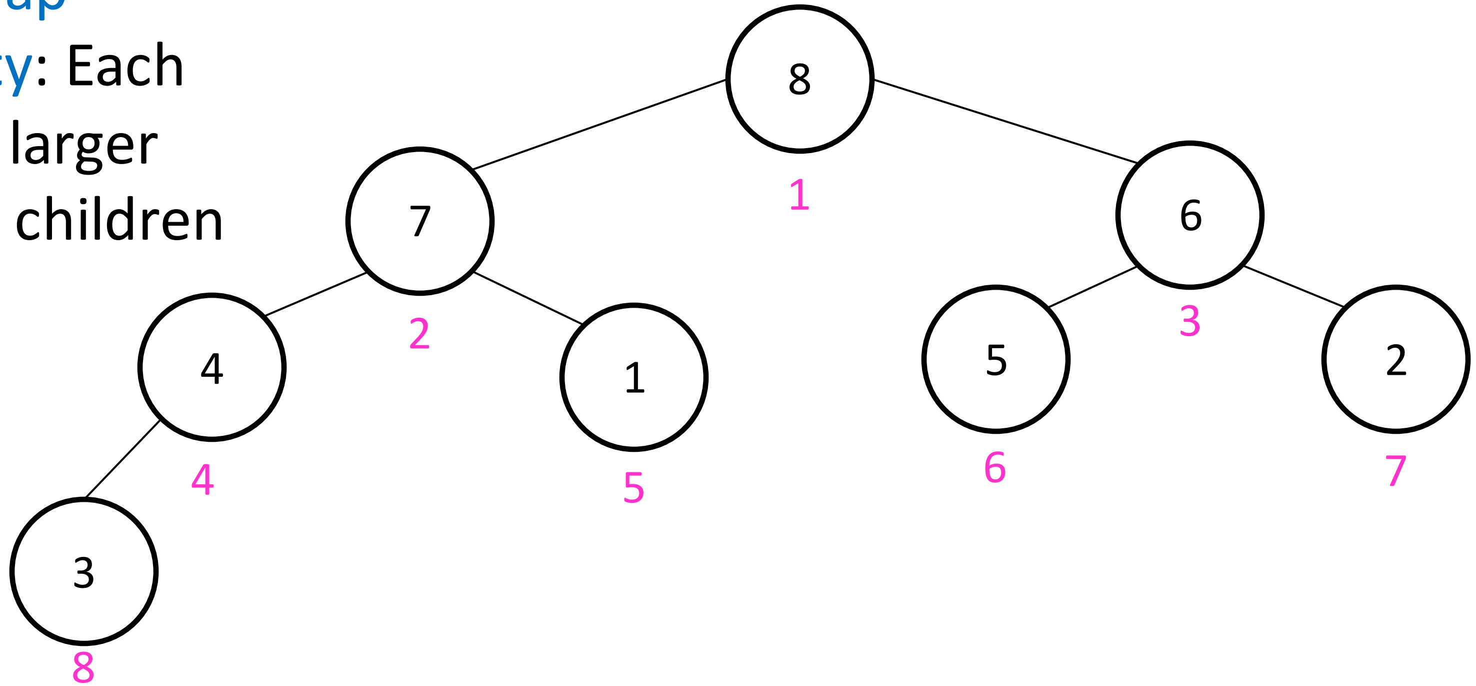


In Place Heap Sort

- Idea: When removing an element from the heap, move it to the (now unoccupied) end of the list

	8	7	6	4	1	5	2	3	9	10
0	1	2	3	4	5	6	7	8	9	10

Max Heap Property: Each node is larger than its children

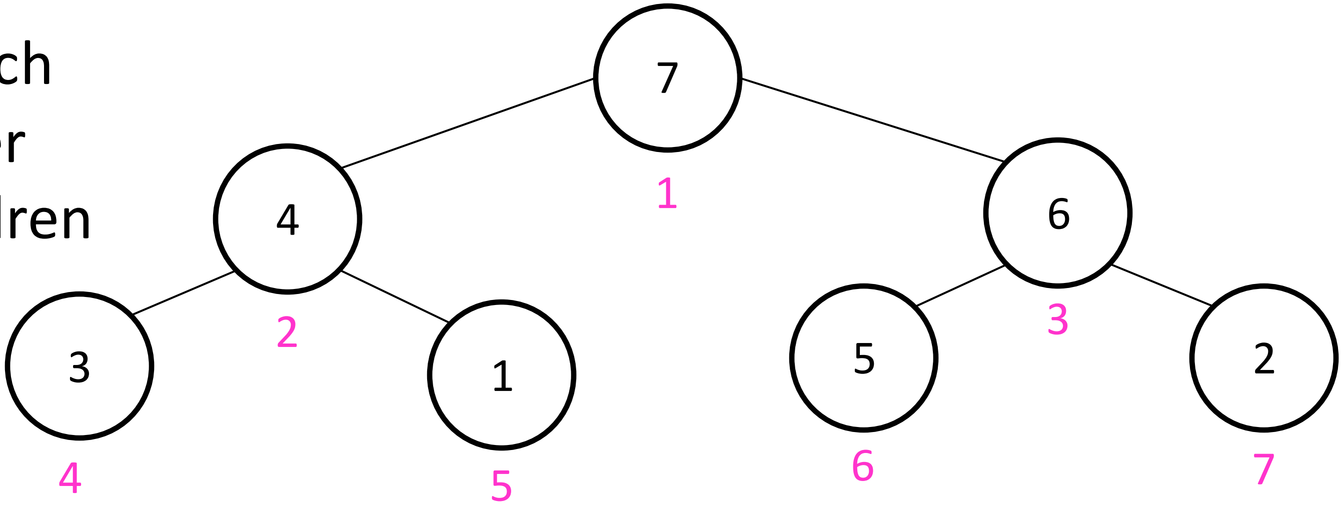


In Place Heap Sort

- Idea: When removing an element from the heap, move it to the (now unoccupied) end of the list

	7	4	6	3	1	5	2	8	9	10
0	1	2	3	4	5	6	7	8	9	10

Max Heap Property: Each node is larger than its children



Heap Sort

- Idea: Build a Heap, repeatedly extract max element from the heap to build sorted list Right-to-Left

Run Time?

$\Theta(n \log n)$

Constants worse than Quick Sort

In Place?	Adaptive?	Stable?	Parallelizable?
Yes!	No	No	No

Sorting in Linear Time

- Cannot be comparison-based
- Need to make some sort of assumption about the contents of the list
  - Small number of unique values
  - Small range of values
  - Etc.

Counting Sort

- Idea: Count how many things are less than each element

$L =$

3	6	6	1	3	4	1	6
1	2	3	4	5	6	7	8

1. Range is $[1, k]$ (here $[1,6]$)
   make an array C of size k
   populate with counts of each value

For i in L:
$++C[L[i]]$

$C =$

2	0	2	1	0	3
1	2	3	4	5	6

running sum

2. Take "running sum" of C
   to count things less than each value

For $i = 1$ to $\text{len}(C)$:
$C[i] = C[i-1] + C[i]$

$C =$

2	2	4	5	5	8
1	2	3	4	5	6

To sort: last item of value 3 goes at index 4

Counting Sort

- Idea: Count how many things are less than each element

$L =$

3	6	6	1	3	4	1	6
1	2	3	4	5	6	7	8

$C =$

2	2	4	5	5	7
1	2	3	4	5	6

Last item of value 6 goes at index 8

For each element of L (last to first):
Use C to find its proper place in B
Decrement that position of C

For $i = \text{len}(L)$ downto 1:

$$B\left[C\left[L[i]\right]\right] = L[i]$$

$$C\left[L[i]\right] = C\left[L[i]\right] - 1$$

$B =$

							6
1	2	3	4	5	6	7	8

Counting Sort

- Idea: Count how many things are less than each element

$L =$

3	6	6	1	3	4	1	6
1	2	3	4	5	6	7	8

$C =$

1	2	4	5	5	7
1	2	3	4	5	6

Last item of value 1 goes at index 2

For each element of L (last to first):
Use C to find its proper place in B
Decrement that position of C

For $i = \text{len}(L)$ downto 1:

$$B\left[C\left[L[i]\right]\right] = L[i]$$

$$C\left[L[i]\right] = C\left[L[i]\right] - 1$$

$B =$

	1						6
1	2	3	4	5	6	7	8

Run Time: $O(n + k)$

Memory: $O(n + k)$

Counting Sort

- Why not always use counting sort?
- For 64-bit numbers, requires an array of length $2^{64} > 10^{19}$
  - 5 GHz CPU will require > 116 years to initialize the array
  - 18 Exabytes of data
    - Total amount of data that Google has (?)

One Exabyte = 10^{18} bytes
1 million terabytes (TB)
1 billion gigabytes (GB)

100,000 x Library of Congress (print)

12 Exabytes

https://en.wikipedia.org/wiki/Utah_Data_Center

Radix Sort

- Idea: Stable sort on each digit, from least significant to most significant

103	801	401	323	255	823	999	101	113	901	555	512	245	800	018	121
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Place each element into a "bucket" according to its 1's place

800	801 401 101 901 121	512	103 323 823 113		255 555 245			018	999
0	1	2	3	4	5	6	7	8	9

Radix Sort

- Idea: Stable sort on each digit, from least significant to most significant

Place each element into a “bucket” according to its 10’s place

800	801 401 101 901 121	512	103 323 823 113		255 555 245			018	999
0	1	2	3	4	5	6	7	8	9

800 801 401 101 901 103	512 113 018	121 323 823		245	255 555				999
0	1	2	3	4	5	6	7	8	9

Radix Sort

- Idea: Stable sort on each digit, from least significant to most significant

0	1	2	3	4	5	6	7	8	9
800 801 401 101 901 103	512 113 018	121 323 823		245	255 555				999

Place each element into a "bucket" according to its 100's place

0	1	2	3	4	5	6	7	8	9
018	101 103 113 121	245 255	323	401	512 555			800 801 823	901 999

Run Time: $O\big(d(n+b)\big)$

d = digits in largest value

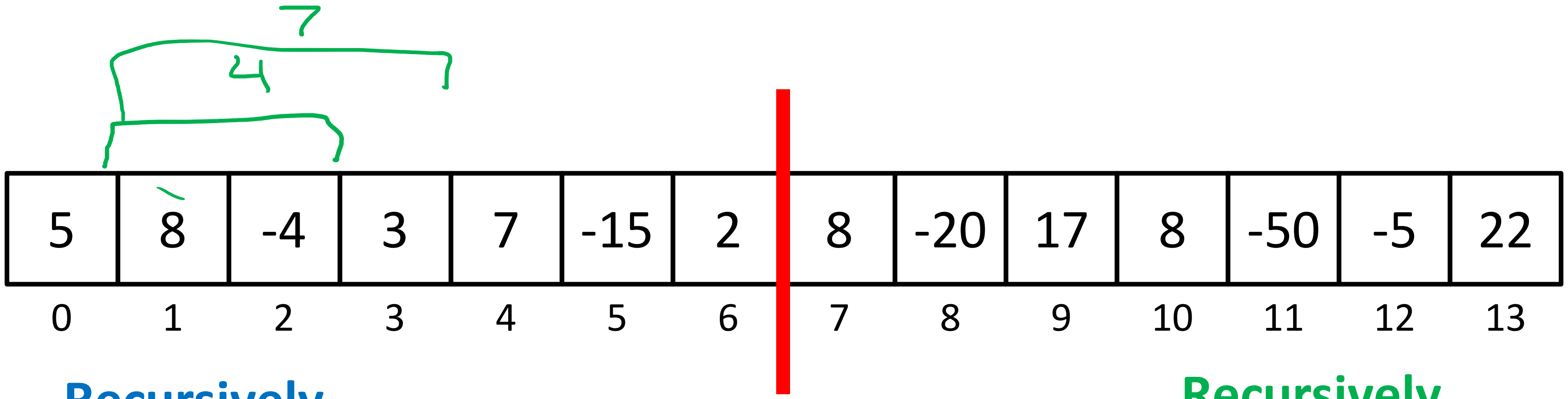
b = base of representation

Maximum Sum Continuous Subarray

The maximum-sum subarray of a given array of integers A is the interval $[a, b]$ such that the sum of all values in the array between a and b inclusive is maximal.

Given an array of n integers (may include both positive and negative values), give a $O(n \log n)$ algorithm for finding the maximum-sum subarray.

Divide and Conquer $\Theta(n \log n)$



Recursively
Solve on Left

Divide in half

Recursively
Solve on Right

Divide and Conquer $\Theta(n \log n)$

Largest sum that ends here + Largest sum that starts here

	6	1	-7	-3	-6	-13	2	8	-12	5	13	-37	-42	-20
	5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Recursively Solve on Left
19

Divide in half

Find Largest sum that spans the cut

Recursively Solve on Right
25

Divide and Conquer $\Theta(n \log n)$

Return the Max of
Left, Right, Center

6	1	-7	-3	-6	-13	2	8	-12	5	13	-37	-42	-20
5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Recursively Solve on Left
19

Divide in half

Find Largest sum that spans the cut
19

Recursively Solve on Right
25

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Divide and Conquer Summary

Typically multiple subproblems.
Typically all roughly the same size.

- Divide
  - Break the list in half
- Conquer
  - Find the best subarrays on the left and right
- Combine
  - Find the best subarray that “spans the divide”
  - I.e. the best subarray that ends at the divide concatenated with the best that starts at the divide

Generic Divide and Conquer Solution

```
def myDCalgo(problem):
    if baseCase(problem):
        solution = solve(problem) #brute force if necessary
        return solution
    subproblems = Divide(problem)
    for sub in subproblems:
        subsolutions.append(myDCalgo(sub))
    solution = Combine(subsolutions)
    return solution
```

MSCS Divide and Conquer $\Theta(n \log n)$

```
def MSCS(list):
    if list.length < 2:
        return list[0]      #list of size 1 the sum is maximal
    {listL, listR} = Divide (list)
    for list in {listL, listR}:
        subSolutions.append(MSCS(list))
    solution = max(solnL, solnR, span(listL, listR))
    return solution
```

Types of “Divide and Conquer”

- Divide and Conquer
  - Break the problem up into several subproblems of roughly equal size, recursively solve
  - E.g. Karatsuba, Closest Pair of Points, Mergesort…
- Decrease and Conquer
  - Break the problem into a single smaller subproblem, recursively solve
  - E.g. Impossible Missions Force (Double Agents), Quickselect, Binary Search

Pattern So Far

- Typically looking to divide the problem by some fraction (½, ¼ the size)
- Not necessarily always the best!
  - Sometimes, we can write faster algorithms by finding unbalanced divides.

Chip and Conquer

- Divide
  - Make a subproblem of all but the last element
- Conquer
  - Find best subarray on the left ($BSL(n-1)$)
  - Find the best subarray ending at the divide ($BED(n-1)$)
- Combine
  - New Best Ending at the Divide:
    - $BED(n) = \max(BED(n-1) + \ arr[n], \ 0)$
  - New best on the left:
    - $BSL(n) = \max\left(BSL(n-1), \ \ BED(n)\right)$

5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Divide

Recursively Solve on Left
25

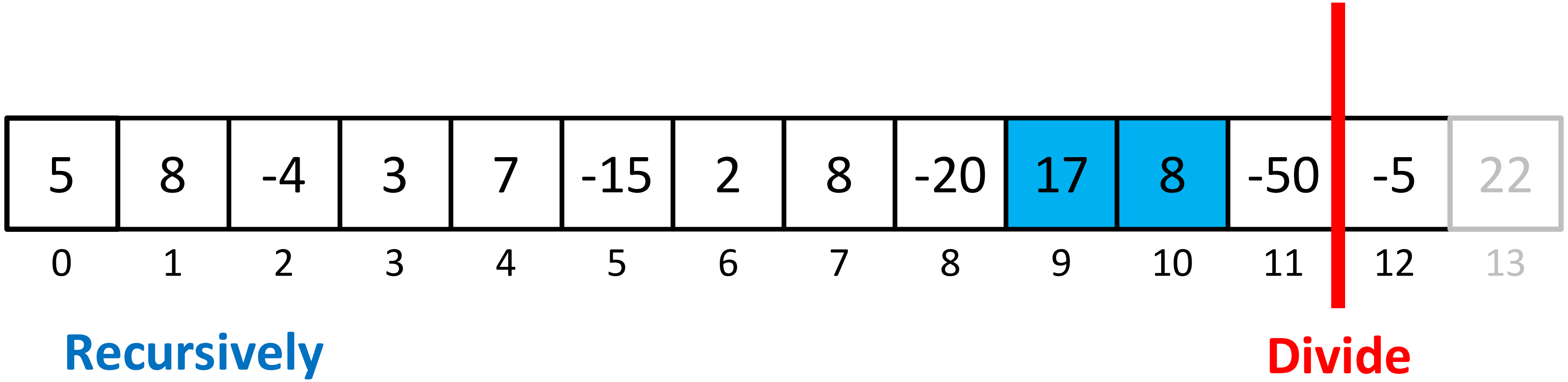
Find Largest sum ending at the cut
22

5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Divide

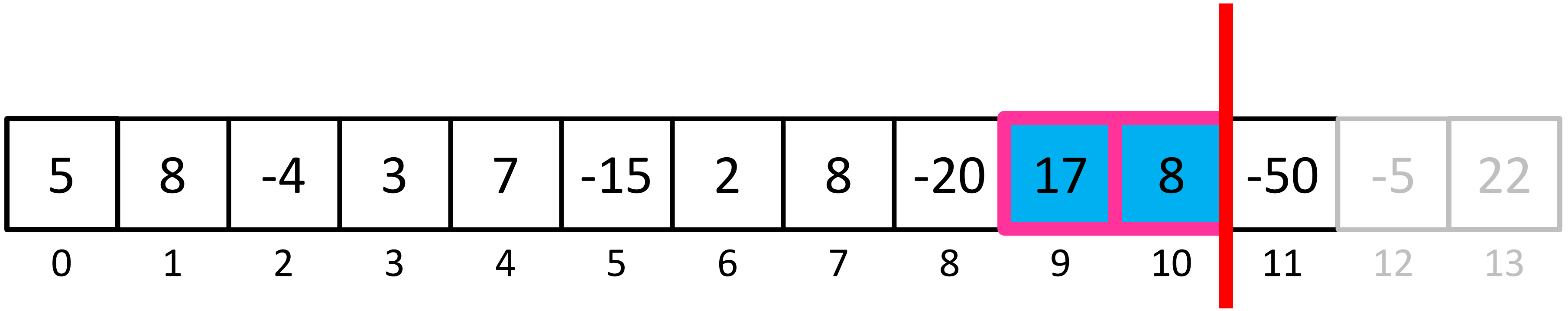
Recursively Solve on Left
25

Find Largest sum ending at the cut
0



Recursively Solve on Left
25

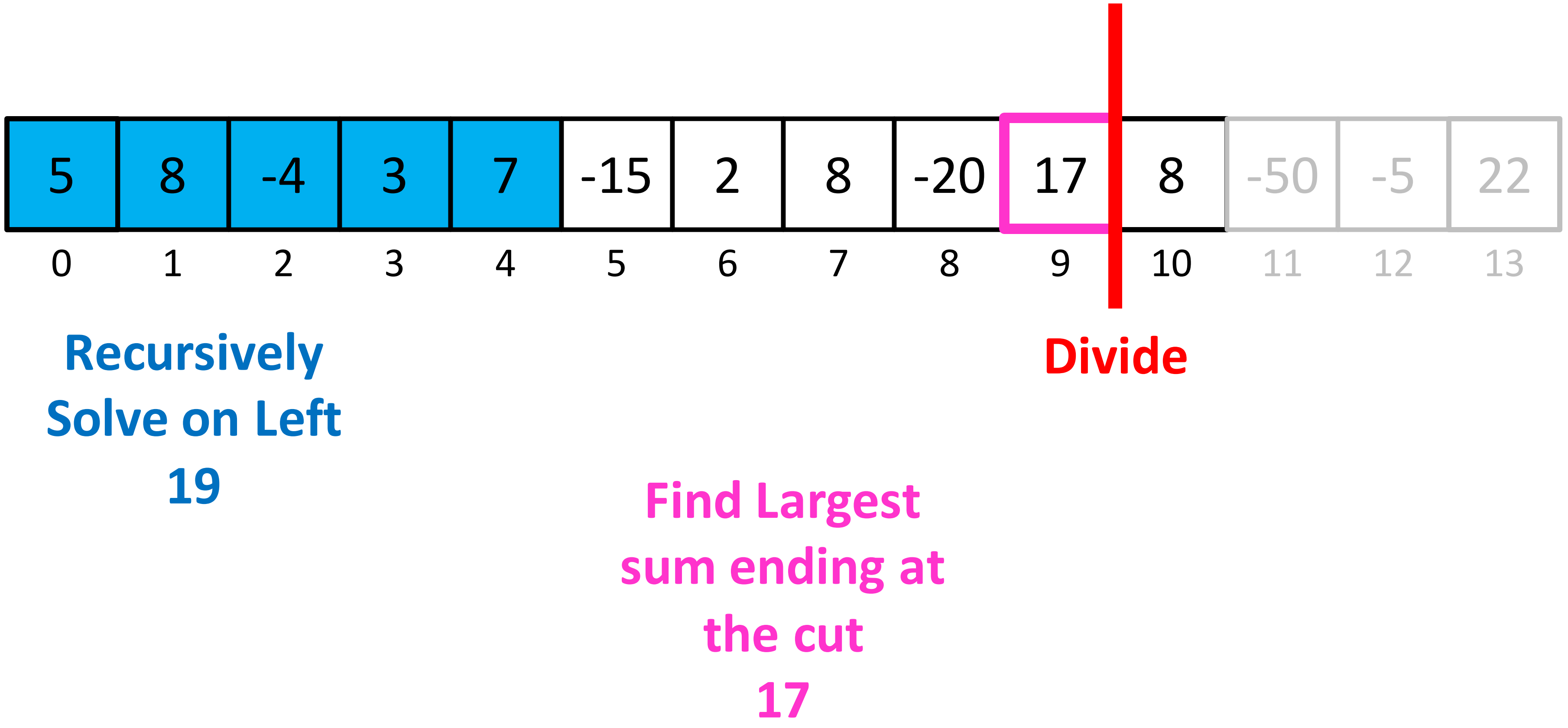
Find Largest sum ending at the cut
0

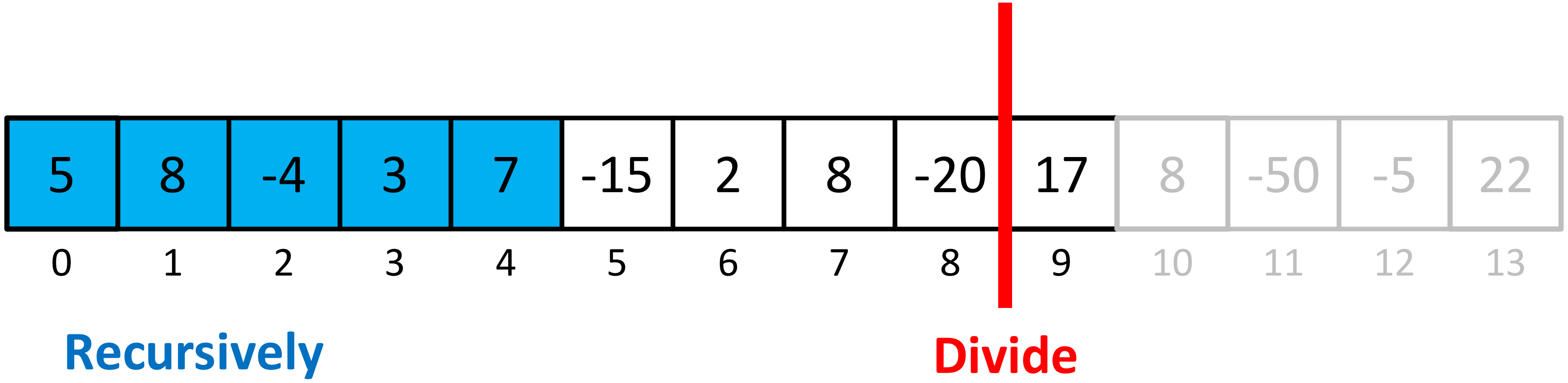


Divide

Recursively Solve on Left
25

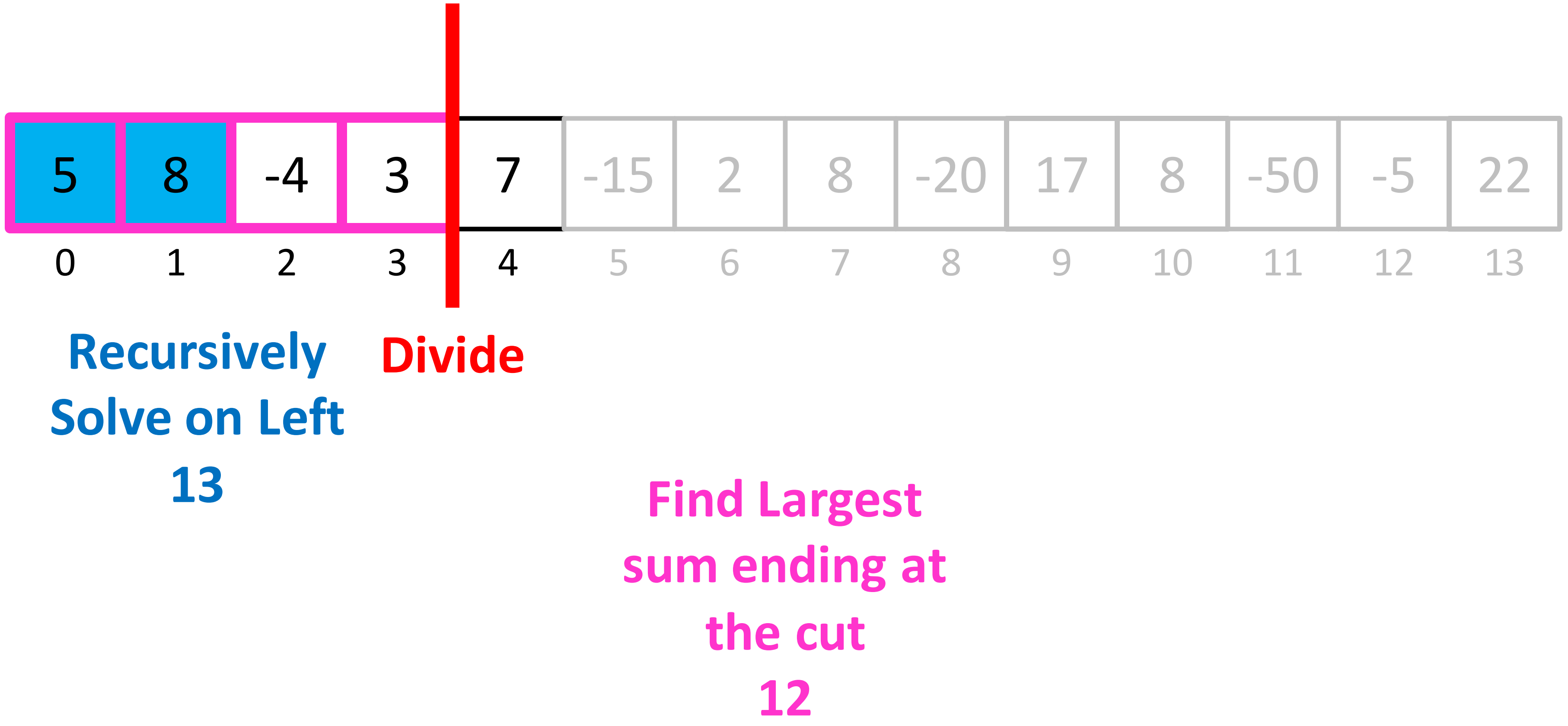
Find Largest sum ending at the cut
25


5
8
-4
3
7
-15
2
8
-20
17
8
-50
-5
22
0
1
2
3
4
5
6
7
8
9
10
11
12
13
Recursively Solve on Left
19
Divide
Find Largest sum ending at the cut
17



Recursively Solve on Left
19

Find Largest sum ending at the cut
0



5
8
-4
3
7
-15
2
8
-20
17
8
-50
-5
22
0
1
2
3
4
5
6
7
8
9
10
11
12
13
Recursively Solve on Left
13
Divide
Find Largest sum ending at the cut
12

Chip and Conquer

- Divide
  - Make a subproblem of all but the last element
- Conquer
  - Find best subarray on the left ($BSL(n-1)$)
  - Find the best subarray ending at the divide ($BED(n-1)$)
- Combine
  - New Best Ending at the Divide:
    - $BED(n) = \max(BED(n-1) + arr[n],\ 0)$
  - New best on the left:
    - $BSL(n) = \max\left(BSL(n-1),\ BED(n)\right)$

Was unbalanced better? YES

- Old:
  - We divided in Half
  - We solved 2 different problems:
    - Find the best overall on BOTH the left/right
    - Find the best which end/start on BOTH the left/right respectively
  - Linear time combine

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T(n) = \Theta(n \log n)$$

- New:
  - We divide by 1, n-1
  - We solve 2 different problems:
    - Find the best overall on the left ONLY
    - Find the best which ends on the left ONLY
  - Constant time combine

$$T(n) = 1T(n-1) + 1$$

$$T(n) = \Theta(n)$$

MSCS Problem - Redux

- Solve in $O(n)$ by increasing the problem size by 1 each time.
- Idea: Only include negative values if the positives on both sides of it are "worth it"

$\Theta(n)$ Solution

5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Begin here

Remember two values:

Best So Far
5

Best ending here
5

$\Theta(n)$ Solution

5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Remember two values:

Best So Far
13

Best ending here
13

$\Theta(n)$ Solution

5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Remember two values:

Best So Far
13

Best ending here
9

$\Theta(n)$ Solution

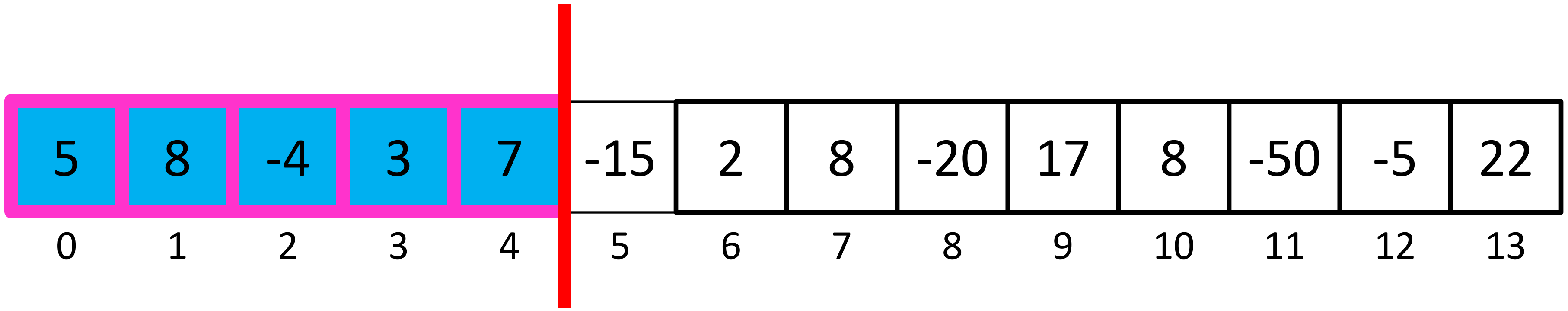
5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Remember two values:

Best So Far
13

Best ending here
12

$\Theta(n)$ Solution

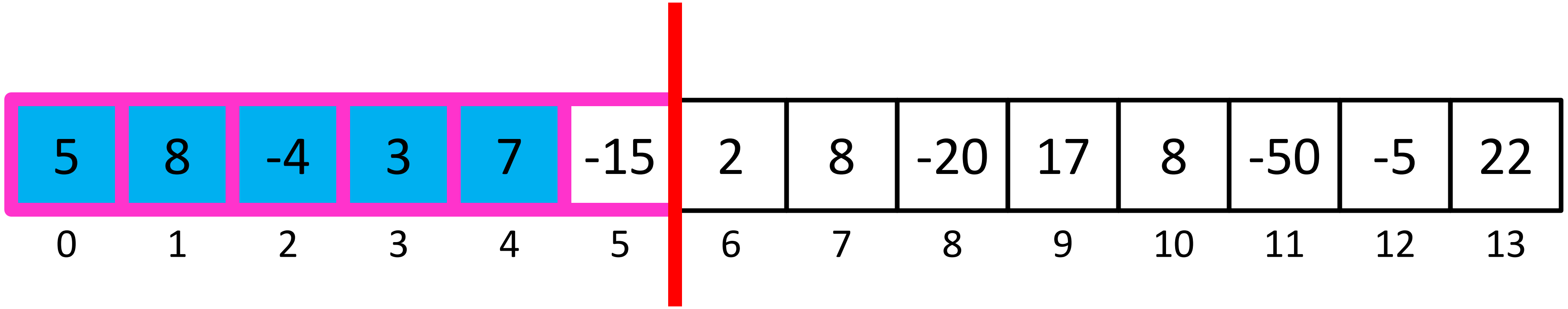


Remember two values:

Best So Far
19

Best ending here
19

$\Theta(n)$ Solution



Remember two values:

Best So Far
19

Best ending here
4

$\Theta(n)$ Solution

5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Remember two values:

Best So Far
19

Best ending here
14

$\Theta(n)$ Solution

5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Remember two values:

Best So Far
19

Best ending here
0

$\Theta(n)$ Solution

5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Remember two values:

Best So Far
19

Best ending here
17

$\Theta(n)$ Solution

5	8	-4	3	7	-15	2	8	-20	17	8	-50	-5	22
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Remember two values:

Best So Far
25

Best ending here
25