Using DFS for Topological Sorting and Strongly Connected Components

CS 4102: Algorithms

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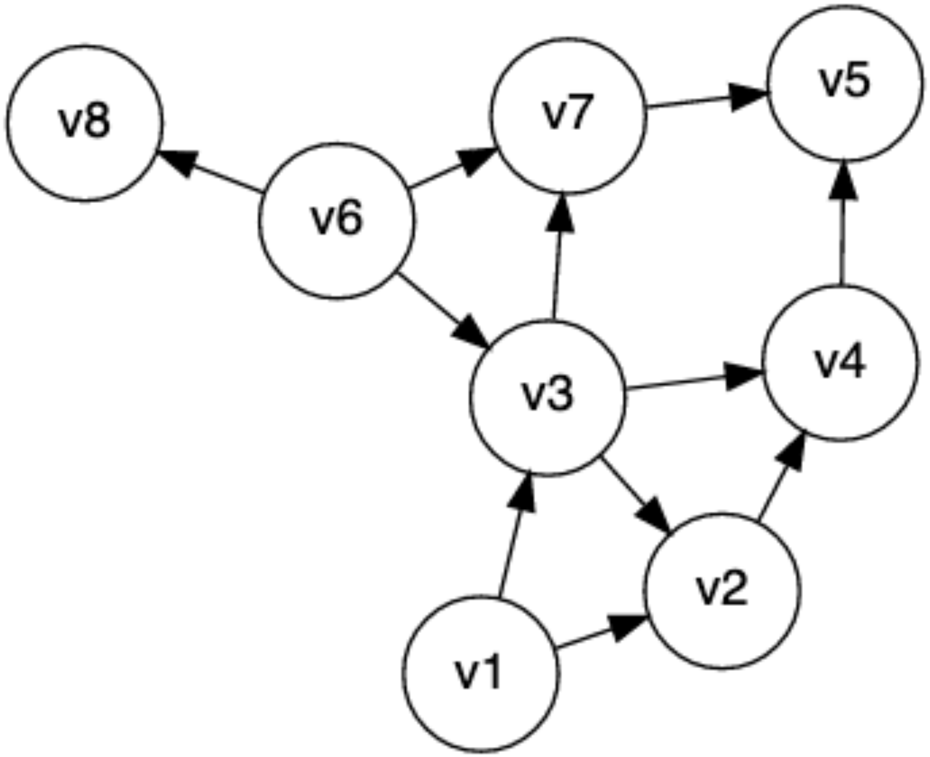
Topological Sorting

Readings: CLRS 22.4

Topological Sort

- Given a ***directed acyclic graph***, construct a linear ordering of the vertices such that if there is an edge from *u* to *v*, then *u* appears before *v* in the ordering.

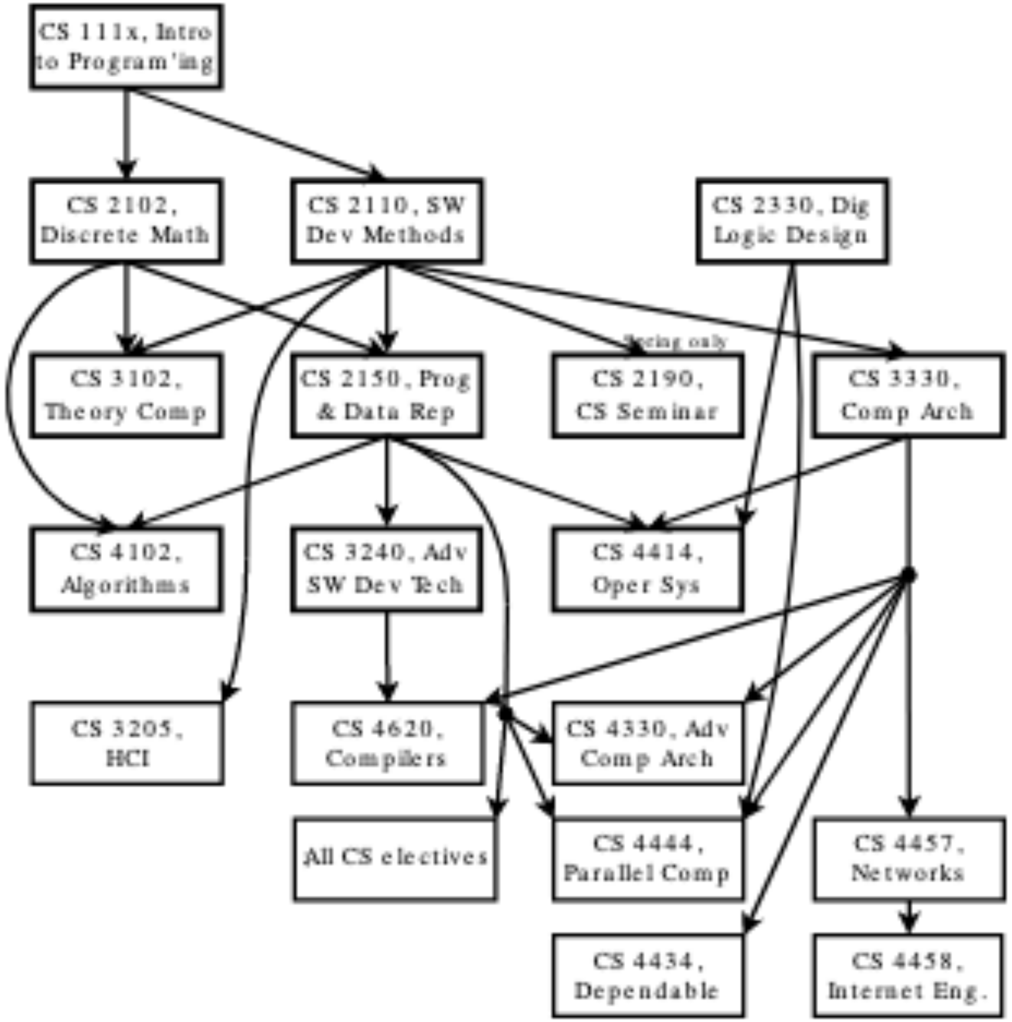


- One valid topological sort is:

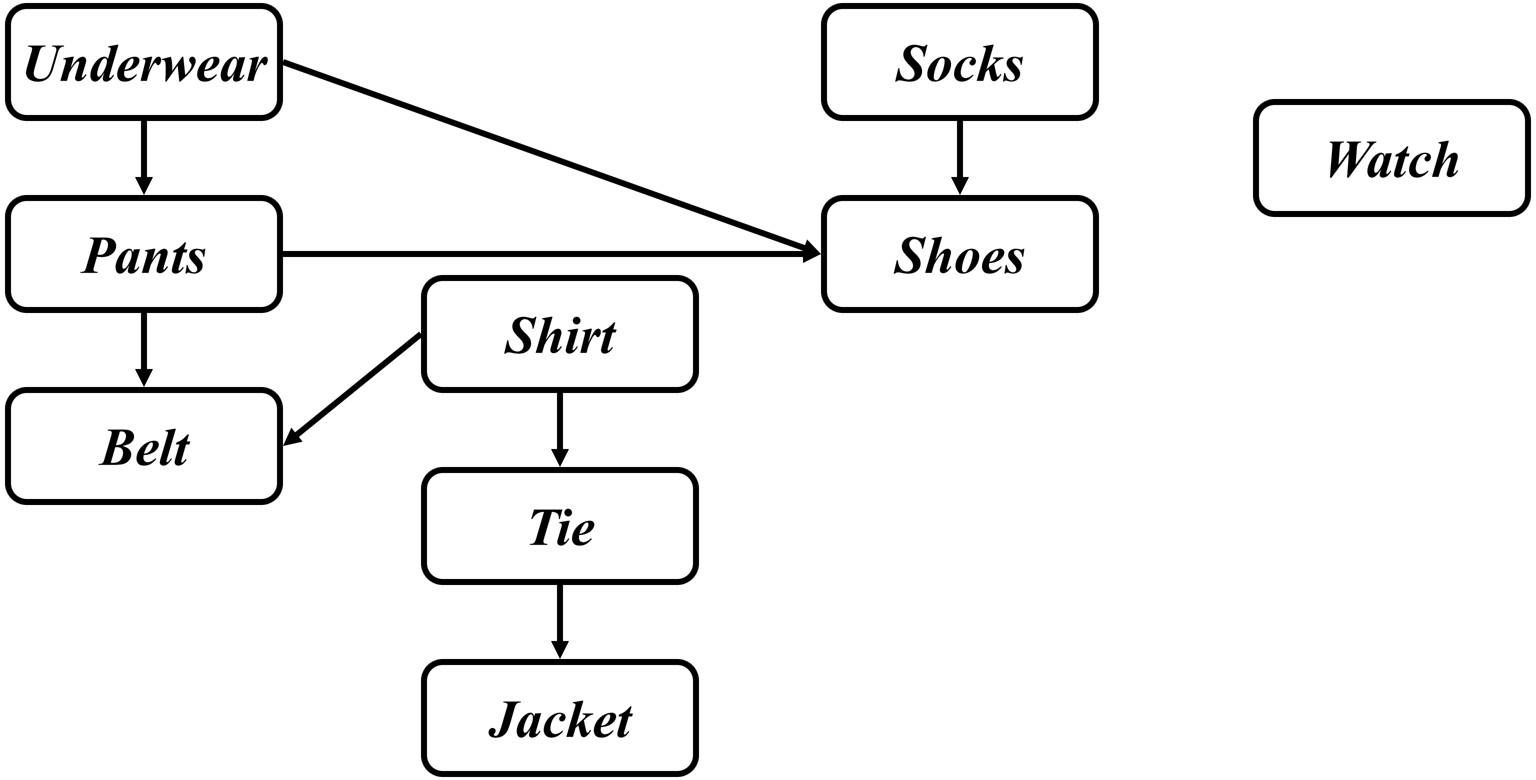
  V1 V6 V8 V3 V2 V7 V4 V5

Topological Sort

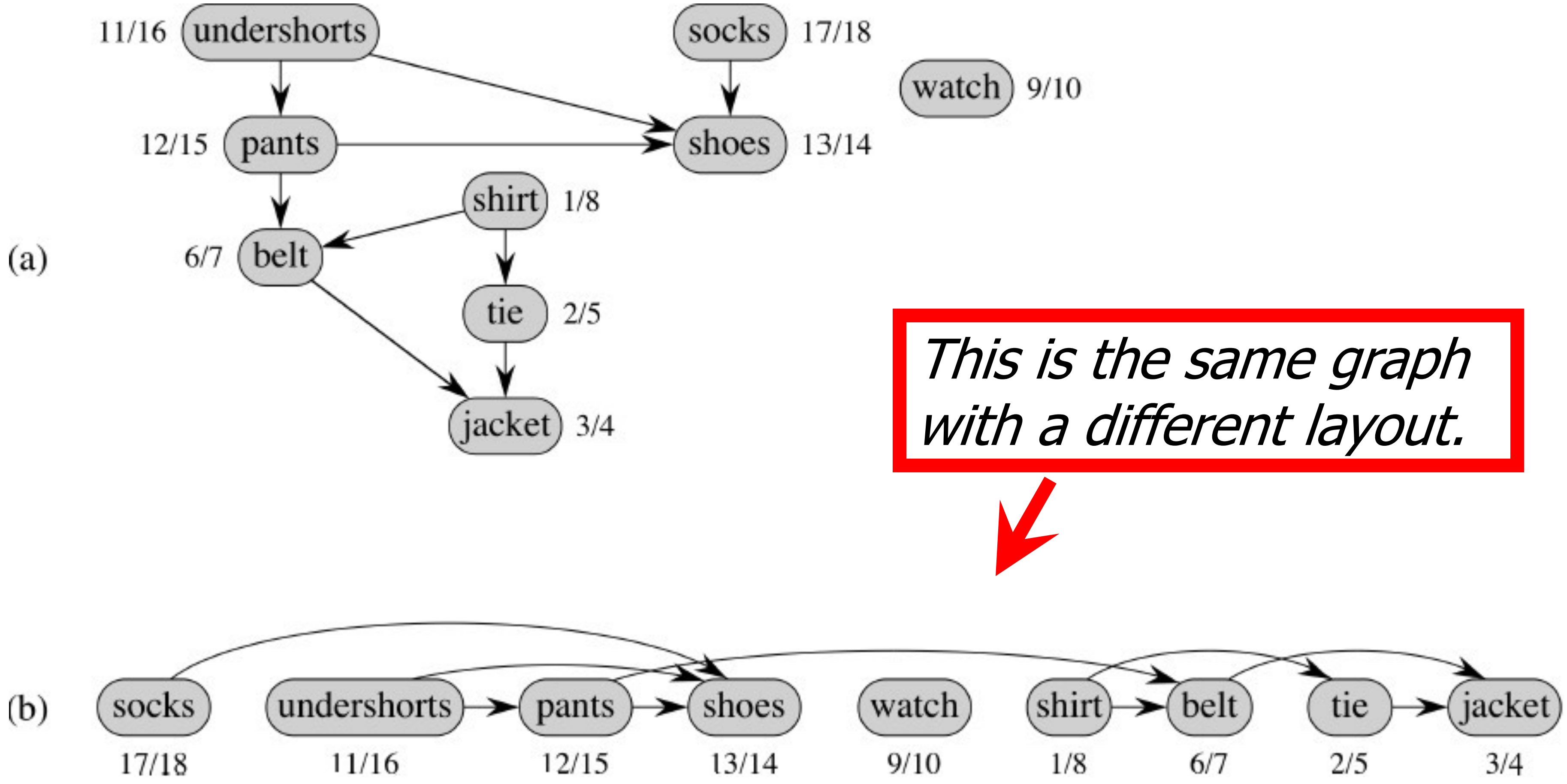
- What are allowable orderings I can take all these CS classes?
  - Note there are many possible orderings
  - Unlike sorting a list



Getting Dressed



We Can Use DFS and Finish Times



Topologically sorted vertices appear in reverse order of their finish times!

Topological Sort Algorithm

- Strategy: modify the two DFS functions so that they order nodes by finish-time in reverse order. This slide: modified version of DFS “Sweep”.

```
DFS_sweep(G)
toposort-list = [ ] // empty list
for each vertex u in G.V
        u.color = WHITE
        u.π = NIL
time = 0
for each vertex u in G.V
        if u.color == WHITE  // if unseen
 DFS-VISIT(G, u)  // explore paths out of u
// toposort-list contains the result
```

Topological Sort Algorithm

```
DFS-VISIT(G, u) // modified to do topological sort
time = time + 1  // white vertex u has just been discovered
u.d = time  // discovery time of u
u.color = GRAY  // mark as seen
for each v in G.Adj[u]  // explore edge (u, v)
    if v.color == WHITE   // if unseen
        v.π = u
        DFS-VISIT(G, v)  // explore paths out of v (i.e., go "deeper")
u.color = BLACK  // u is finished
time = time + 1
u.f = time  // finish time of u
toposort-list.prepend(u)
```

Forward vs. Reverse

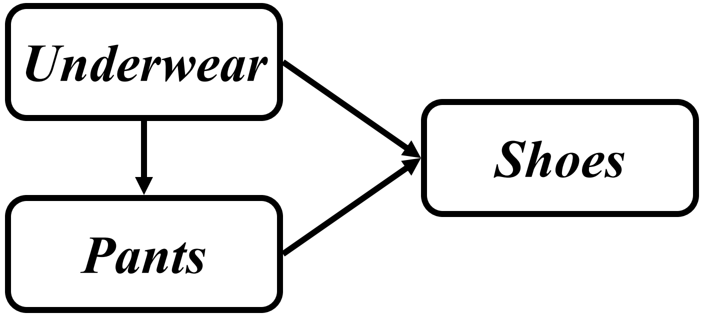
- Topological sort is a type of sort
  - Implies an ordering
  - Can sort backwards, of course

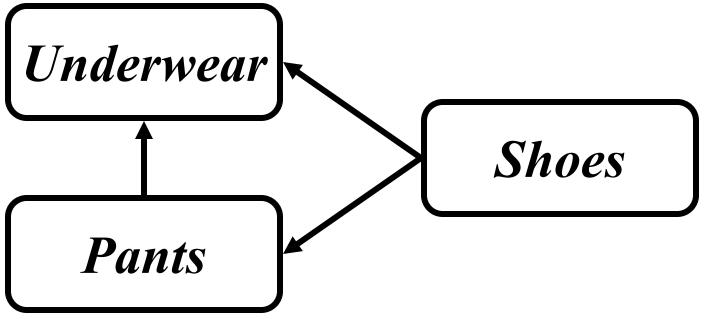
- Forward topological order
  - If edge **vw** in graph, then topo[**v**] < topo[**w**]
- Reverse topological order
  - If edge **vw** in graph, then topo[**v**] > topo[**w**]

- And, every directed graph has a transpose, which means… (see next slide)

What's an Edge Mean?

- What does our graph model?
  - Edge **uv** means do **u** first, then **v**. Or, …
  - Edge **uv** means task **u** depends on v (i.e. **v** must be done first)





  – The latter is called a dependency graph
  – "forward in time" vs. "depend on this one"
- Big deal? No, we can order vertices in reverse topological order if needed

Strongly Connected Components in a Digraph

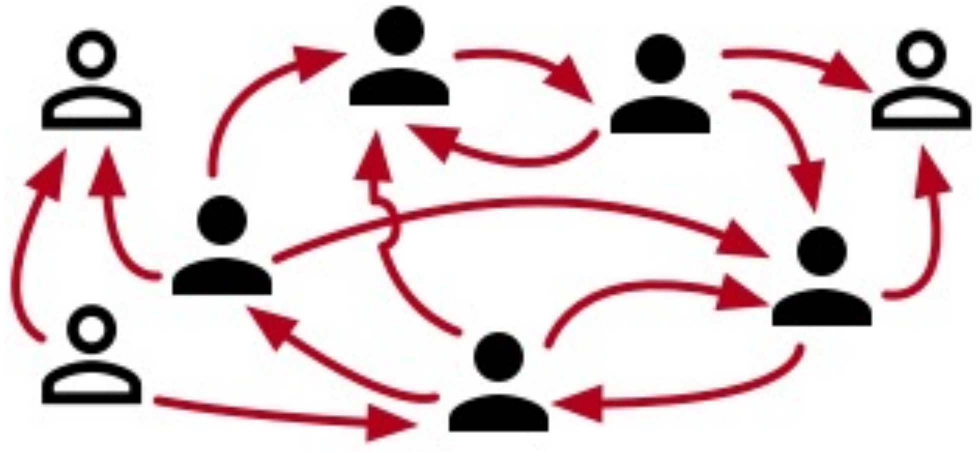
Readings: CLRS 22.5, but you can ignore the proof-y parts

Strongly Connected Components (SCCs)

- In a digraph, Strongly Connected Components (SCCs) are subgraphs where all vertices in each SCC are reachable from one another
  - Thus vertices in an SCC are on a directed cycle
  - Any vertex not on a directed cycle is an SCC all by itself
- Common need: decompose a digraph into its SCCs
  - Perhaps then operate on each, combine results based on connections between SCCs

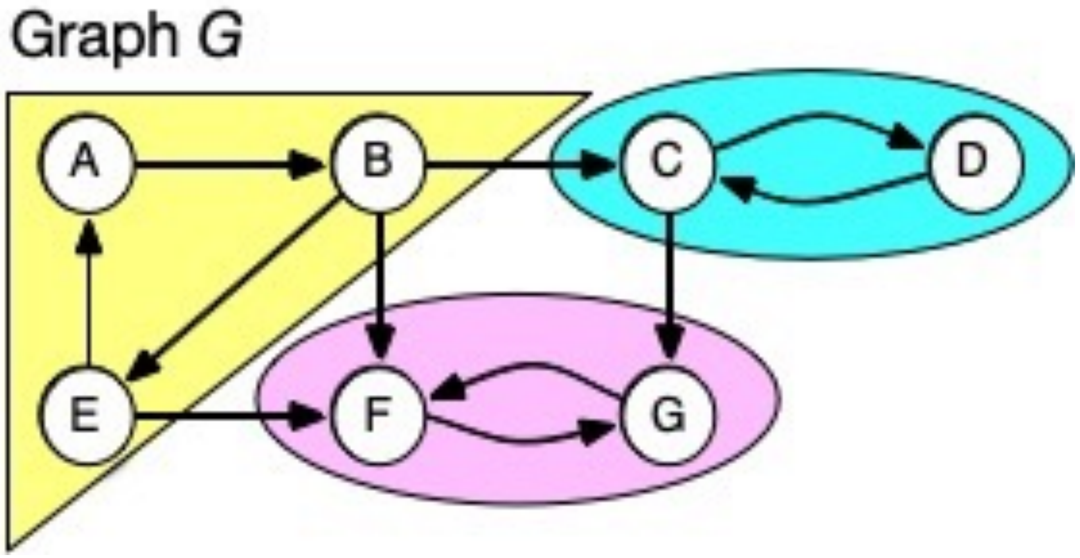
Real-world Example: Social Networks

- Model a social network of users
  - Directed edge *u->v* means *u* follows *v*
- We want to identify a group of users who follow each other
  - Maybe not directly
  - OK if it's indirect, i.e. if there's a path connecting any pair in the group
- In this example, the group of solid-colored users is an SCC
- Note: if all pairs had to follow each other, we call this a *clique*



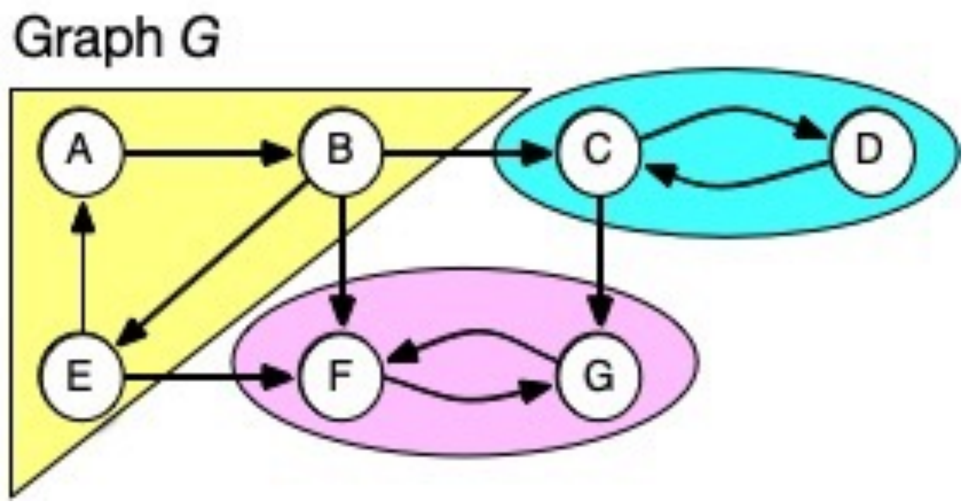
SCC Example

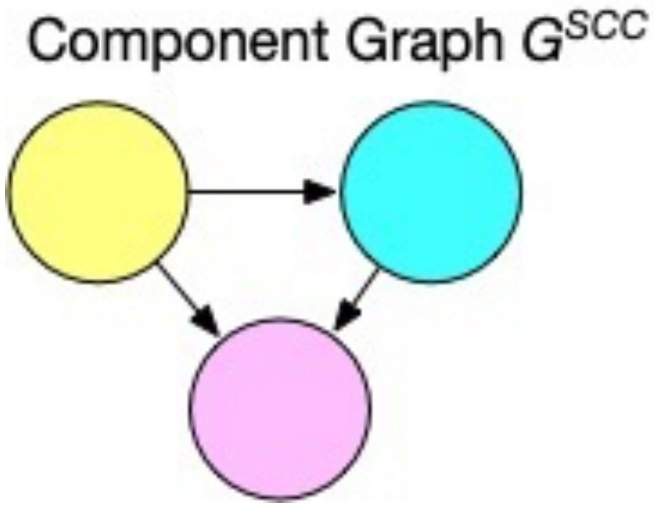
- Example: digraph below has 3 SCCs
  - Note here each SCC has a cycle. (Possible to have a single-node SCC.)
  - Note connections to other SCCs, but no path leaves a SCC and comes back
  - Note there's a unique set of SCCs for a given digraph



Component Graph

- Sometimes for a problem it's useful to consider digraph G's **component graph,** G^{SCC}
  - It's like we "collapse" each SCC into one node
  - Might need a topological ordering between SCCs





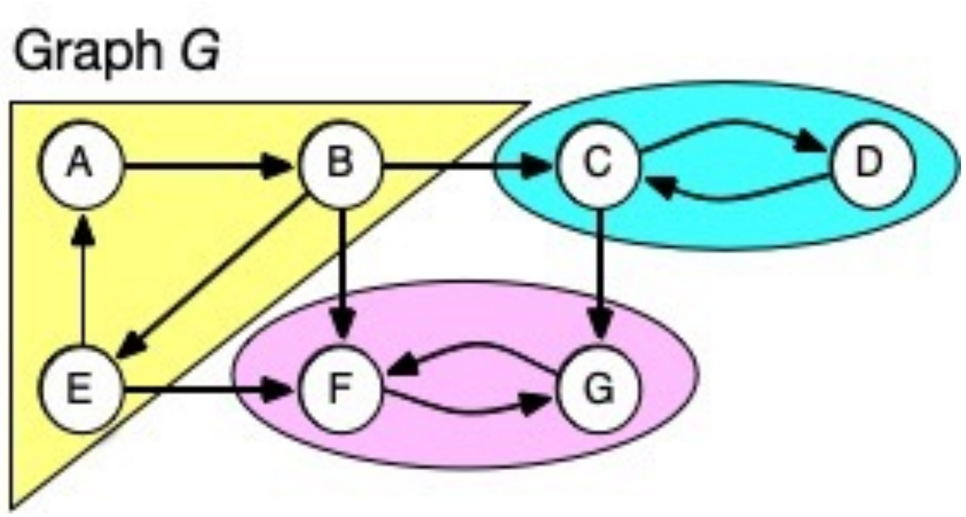
How to Decompose Digraph into SCCs

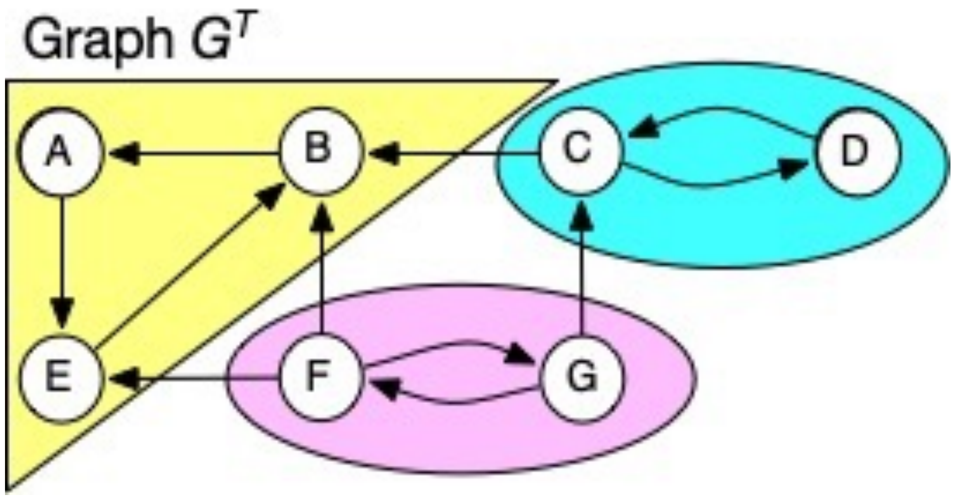
- Several algorithms do this using DFS
- We'll use CLRS's choice (by Kosaraju and Sharir)
- Algorithm works as follows:
  1. Call *DFS-sweep(G)* to find finishing times *u.f* for each vertex *u* in *G*.
  2. Compute G^T, the transpose of digraph *G*.

     (Reminder: transpose means same nodes, edges reversed.)
  3. Call *DFS-sweep*(G^T) but do the recursive calls on nodes in the order of decreasing *u.f* from Step 1. (Start with the vertex with largest finish time in G's DFS tree,…)
  4. The DFS forest produced in Step 3 is the set of SCCs

Why Do We Care about the Transpose?

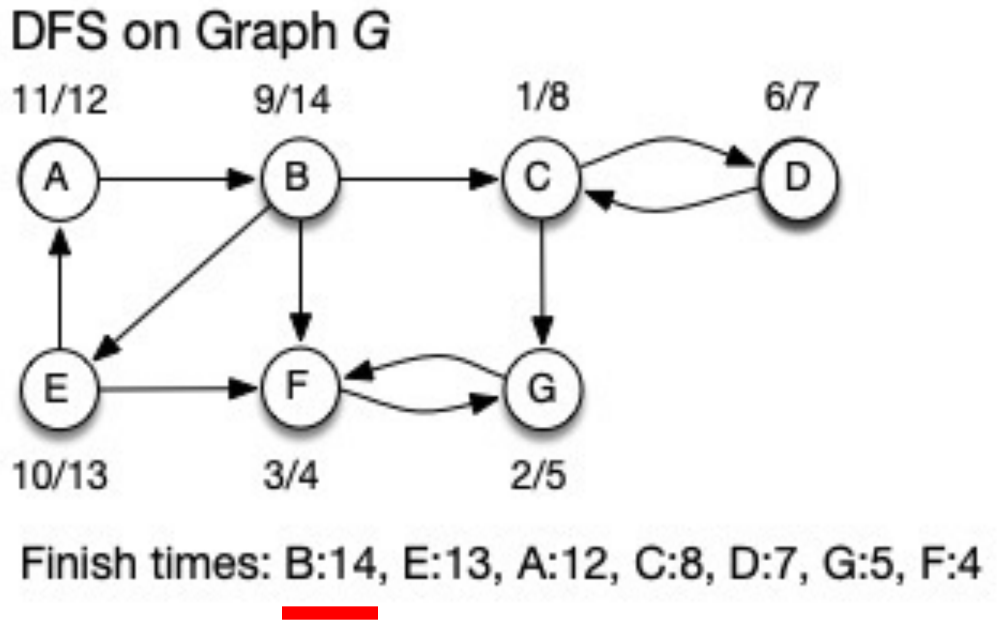
- If we call DFS on a node in an SCC, it will visit all nodes in that SCC
  - But it could leave the SCC and find other nodes ☹
  - Could we prevent that somehow?
- Note that a digraph and its transpose have the same SCCs
  - Maybe we can use the fact that edge-directions are reversed in G^T to stop DFS from leaving an SCC?
  - But this depends on the order you choose vertices to do *DFS-sweep()* in G^T

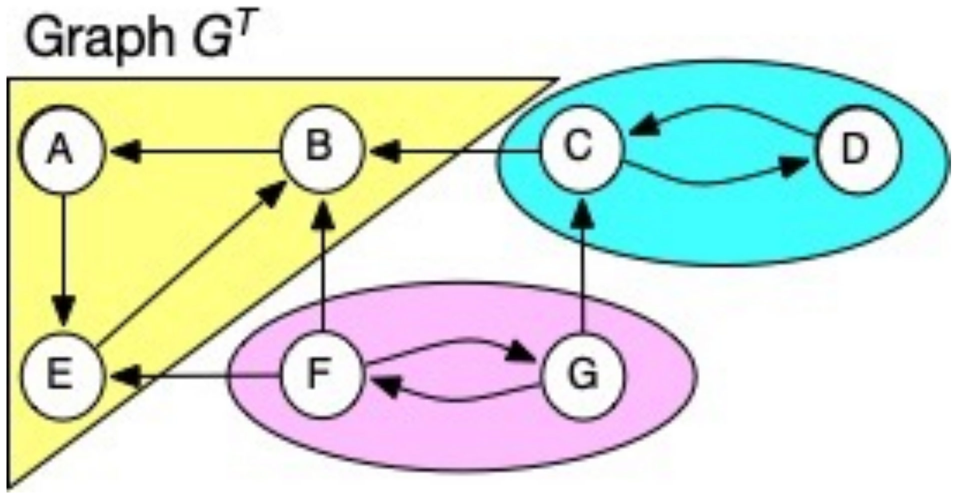




Why Do We Care About Finish Times?

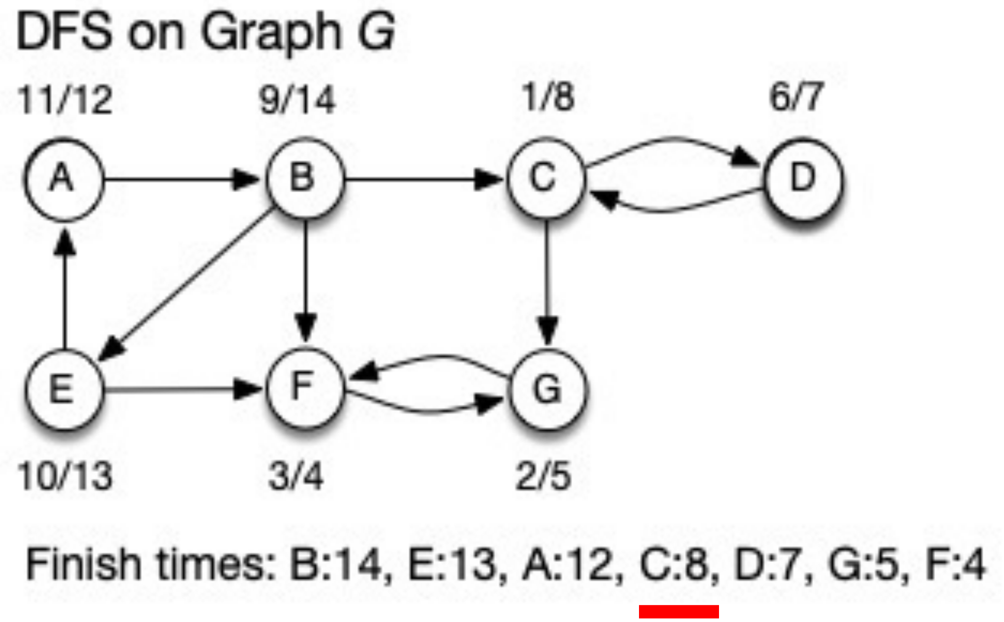
- Our algorithm first finds DFS finish times <u>in G</u>
- Then calls recursive DFS <u>on transpose G^T</u> from vertex with largest finish time (here, B)
  - Reversed edges in G^T stop it visiting nodes in other SCCs

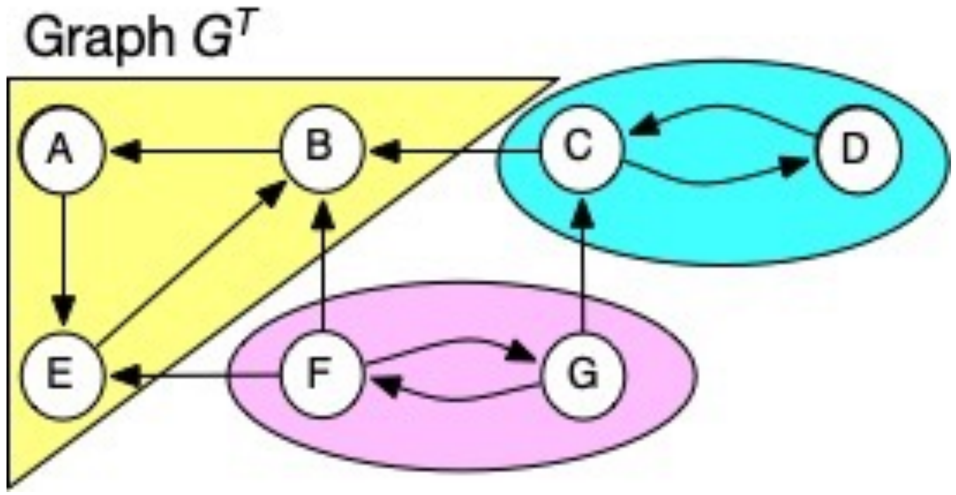




Why Do We Care About Finish Times?

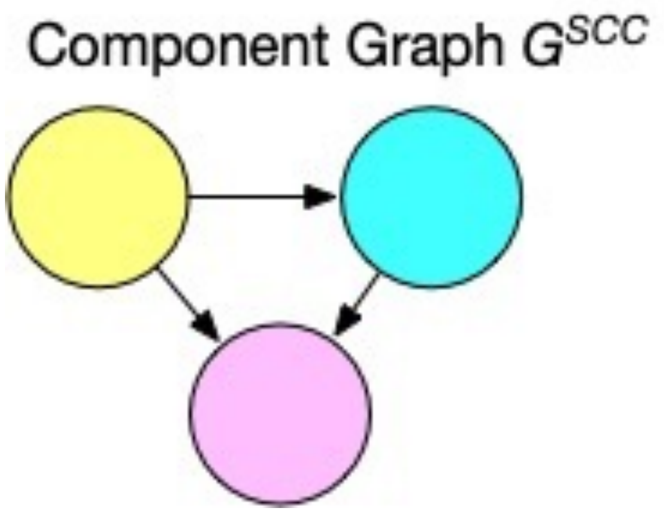
- After recursive DFS <u>on transpose G^T</u> finds SCC containing B, next DFS will start from C
  - Nodes in previously found SCC(s) have been visited
  - Reversed edges in G^T stop it visiting nodes in SCCs yet to be found

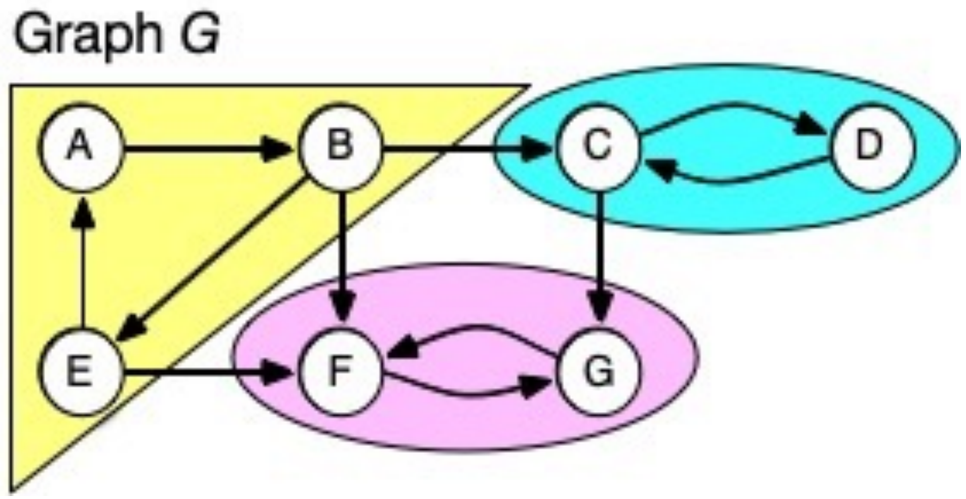




Ties to Topological Sorting

- Formal proof of correctness in CLRS, but hopefully from previous slides you're convinced it works!
- Note how the use of finish times makes this seem like topological sort. And it is, if you think of topological ordering for G^{SCC}
  - Cycles in G, but no cycles in G^{SCC} so we could sort that
  - Topological sort controls the order we do things, and DFS finds all the reachable nodes in an SCC





Final Thoughts

- There are many interesting problems involving digraphs and DAGs
- They can model real-world situations
  - Dependencies, network flows, …
- DFS is often a valuable strategy to tackle such problems
  - For DAGs, not interested in back-edges, since DAGs are acyclic
  - Ordering, reachability from DFS can be useful