CS4102 Algorithms

Spring 2022

<u>Warm up:</u>

Show that $P = NP$

Today's Keywords

- Reductions
- P vs NP
- NP Hard, NP Completeness
- k-Independent Set
- k-Vertex Cover
- 3SAT
- k-Clique

- CLRS: Ch 34

Homeworks

- Unit C and D Programming Due 5/3
- Unit D Advanced Due 5/3
  - NP Completeness and Reductions
- Unit D Basic Due 5/3 (but no penalty submission through 5/6)

Final Exam

- Tuesday, May 10, 7pm in MEC 205 (our section)

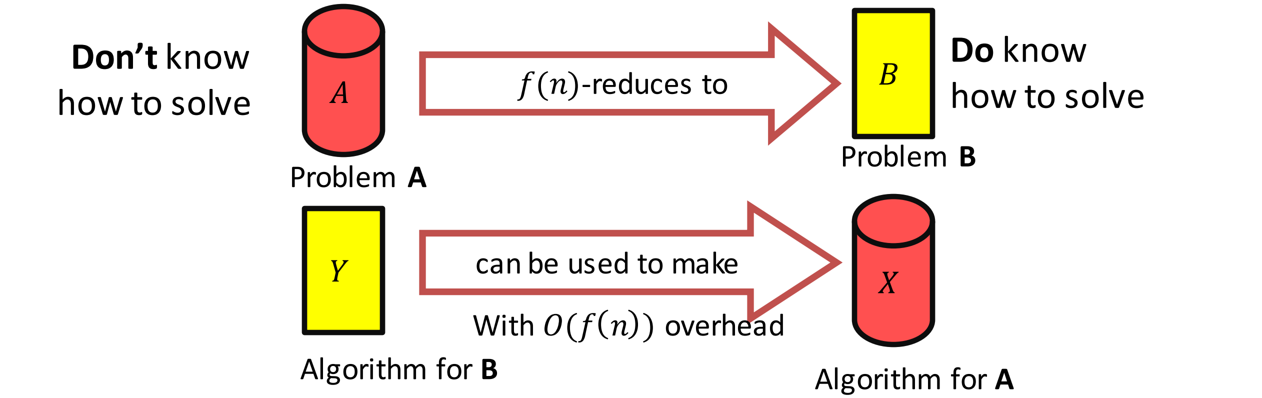
Reductions

- Algorithm technique of supreme ultimate power
- Convert instance of problem A to an instance of Problem B
- Convert solution of problem B back to a solution of problem A

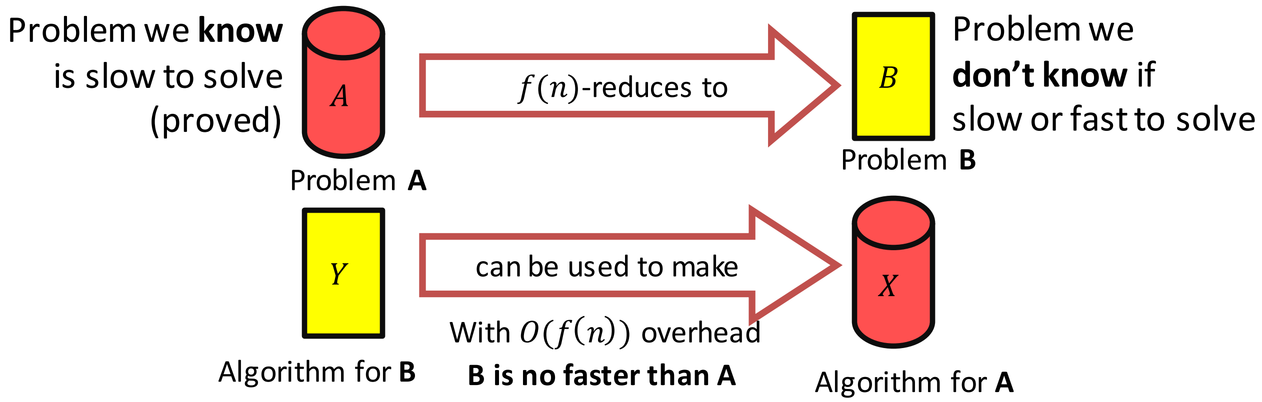
Reductions

Possible uses

- Use solver for B to solve A



- Prove lower bound for B by showing it's as hard as A



MacGyver's Reduction

Problem we don't know how to solve

A

Opening a door

Solution for $\boldsymbol{A}$

Keg cannon battering ram

Aim duct at door, insert keg

Put fire under the Keg

Reduction

Problem we do know how to solve

B

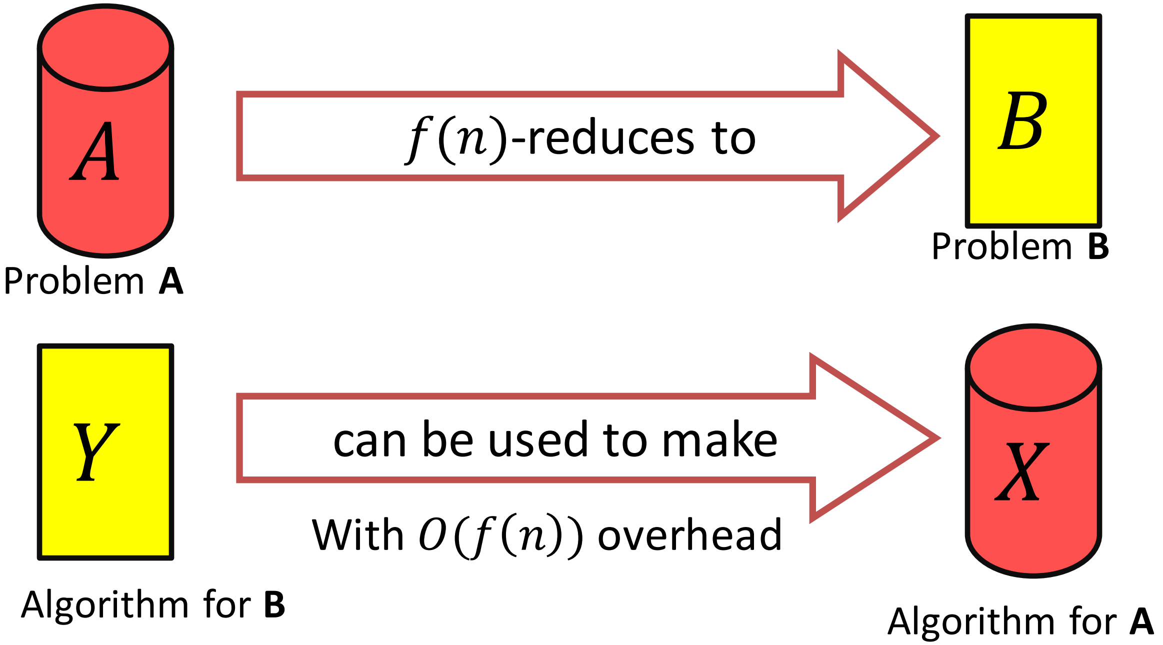
Lighting a fire

How?

Solution for $\boldsymbol{B}$

Alcohol, wood, matches

Reduction Proof Notation



A is not a harder problem than B

$$A \le B$$

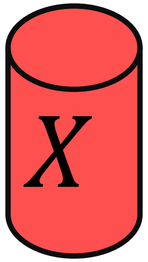
If A requires time $\Omega(f(n))$ time then B also requires $\Omega(f(n))$ time

$$A \le_{f(n)} B$$

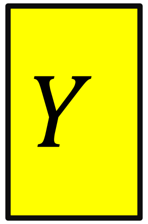
Or we could have solved A faster using B's solver!

Proof of Lower Bound by Reduction

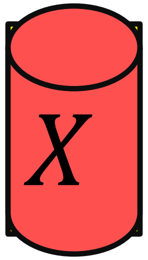
<u>To Show: Y is slow</u>



1. We know X is slow (by a proof)
(e.g., X = some way to open the door)



2. Assume Y is quick [toward contradiction]
(Y = some way to light a fire)



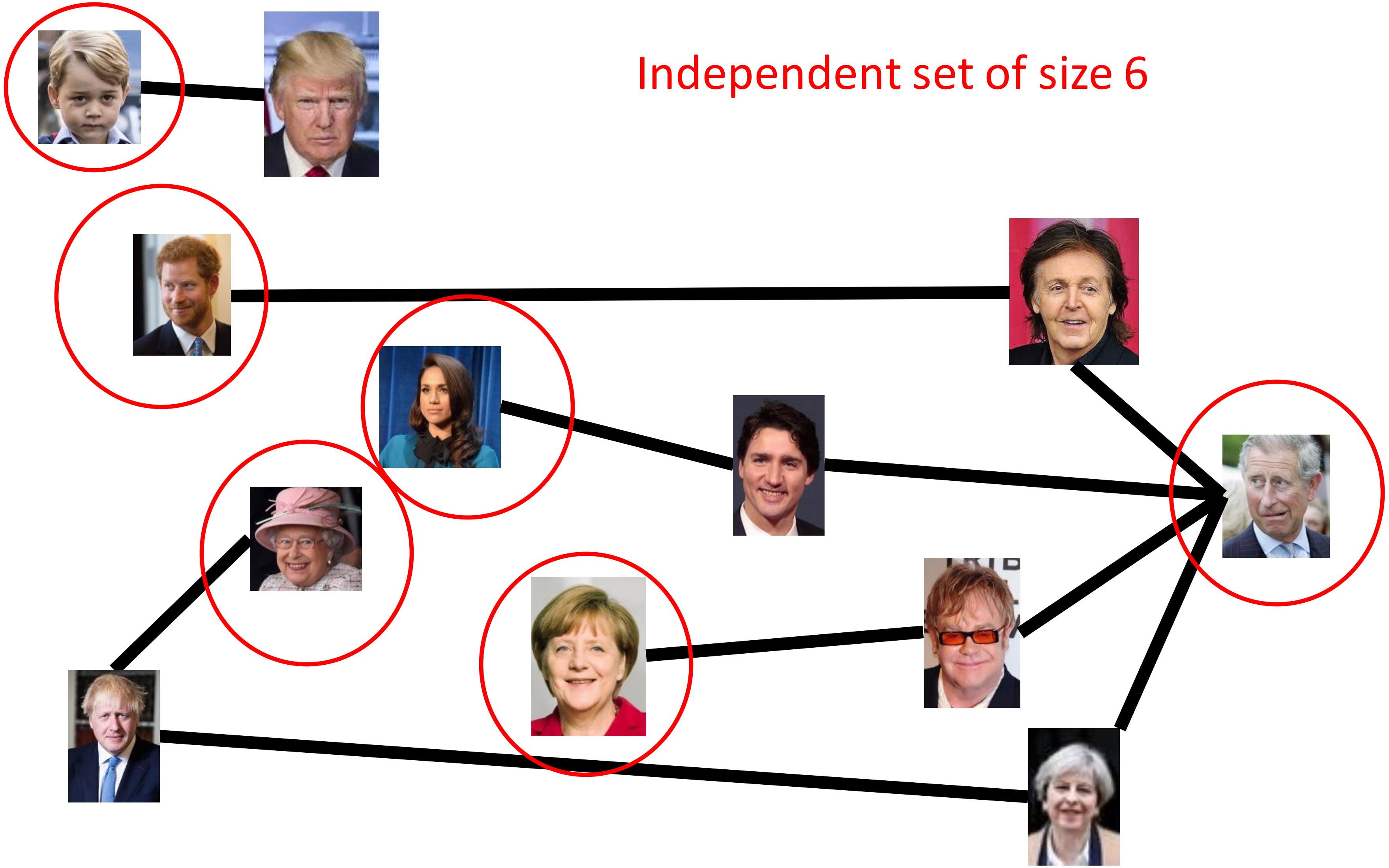
3. Show how to use Y to perform X quickly

4. X is slow, but Y could be used to perform X quickly
conclusion: Y must not actually be quick

Maximum Independent Set

- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Maximum Independent Set Problem: Given a graph $G = (V, E)$ find the maximum independent set S

Example

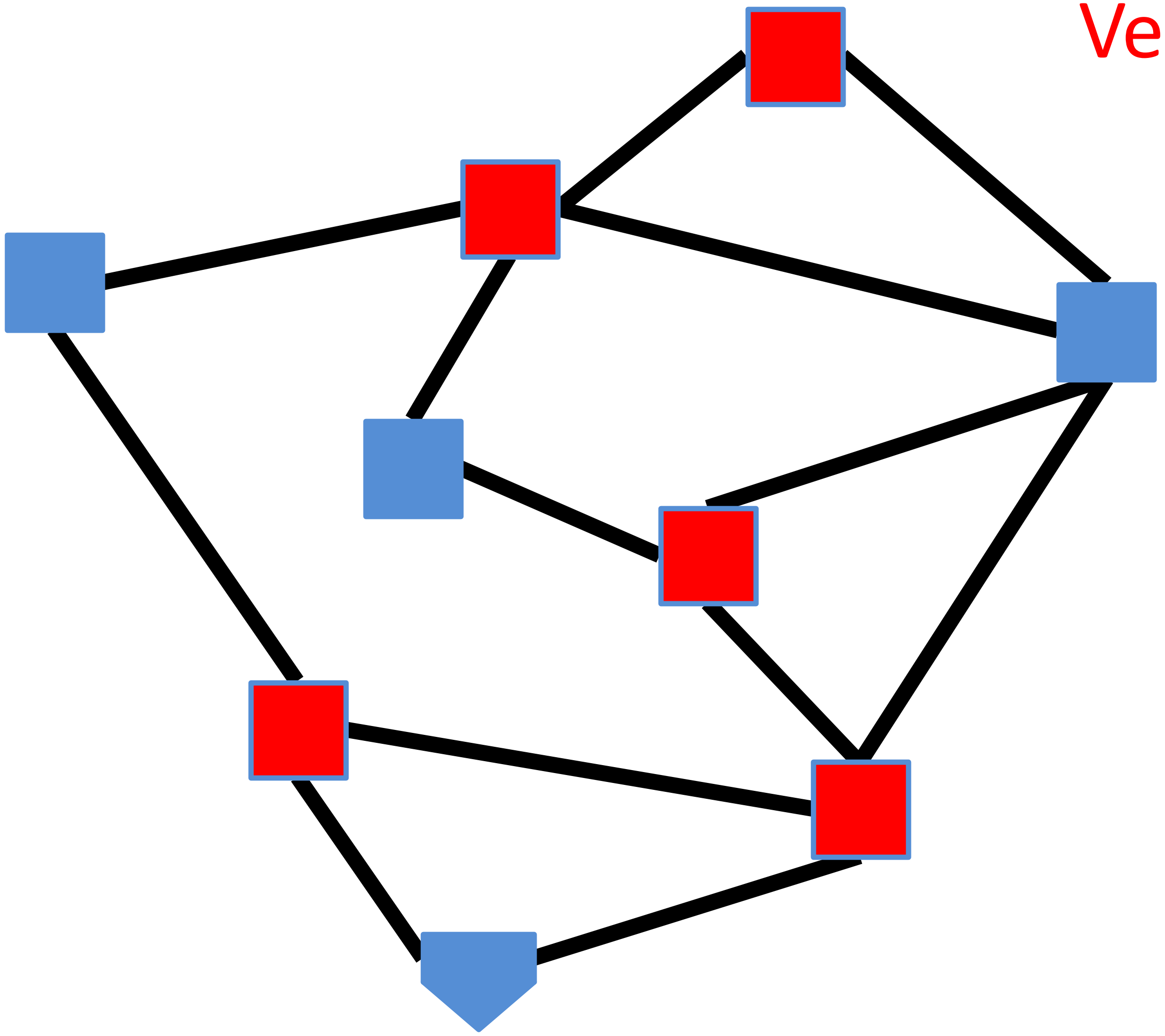


Minimum Vertex Cover

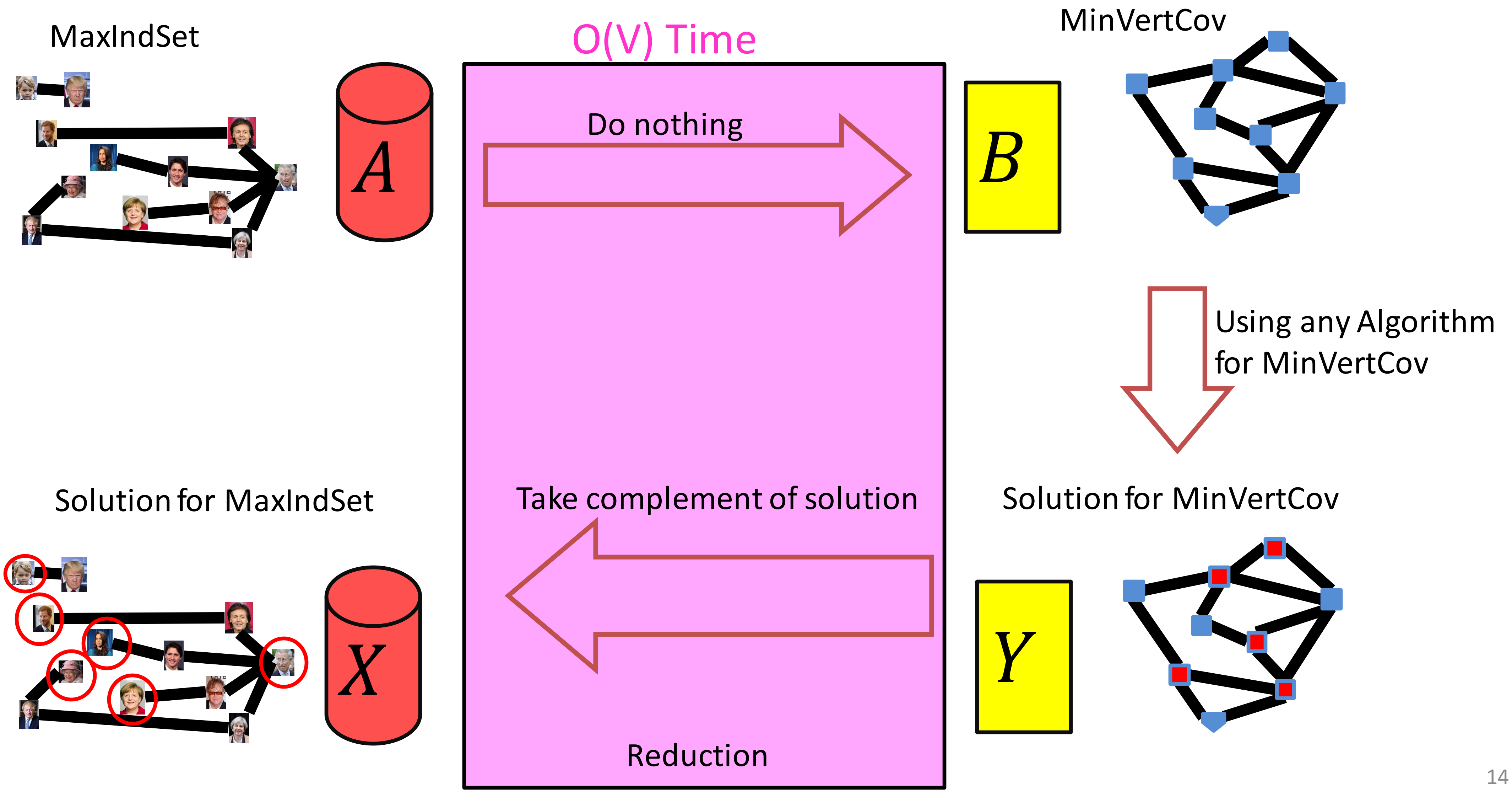
- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Minimum Vertex Cover: Given a graph $G = (V, E)$ find the minimum vertex cover C

Example

Vertex cover of size 5

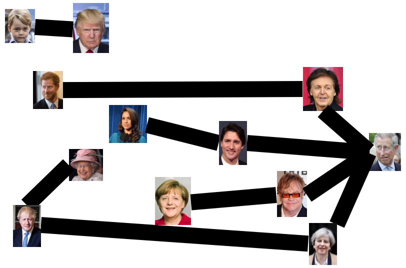


MaxIndSet V-Time Reducible to MinVertCover



MaxIndSet V-Time Reducible to MinVertCov

MaxIndSet



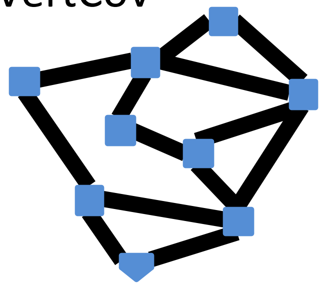
A

O(V) Time

Do nothing

B

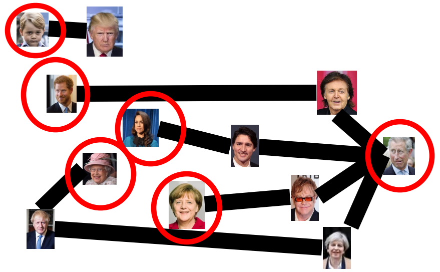
MinVertCov



We needed our proof to show that this works!

Using any Algorithm for MinVertCov

Solution for MaxIndSet



X

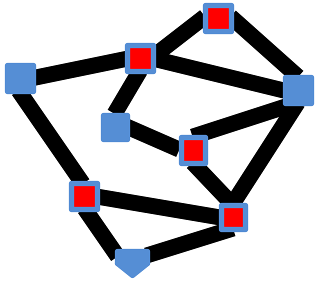
Take complement of solution

If there was a larger independent set, there would have been a smaller vertex cover!

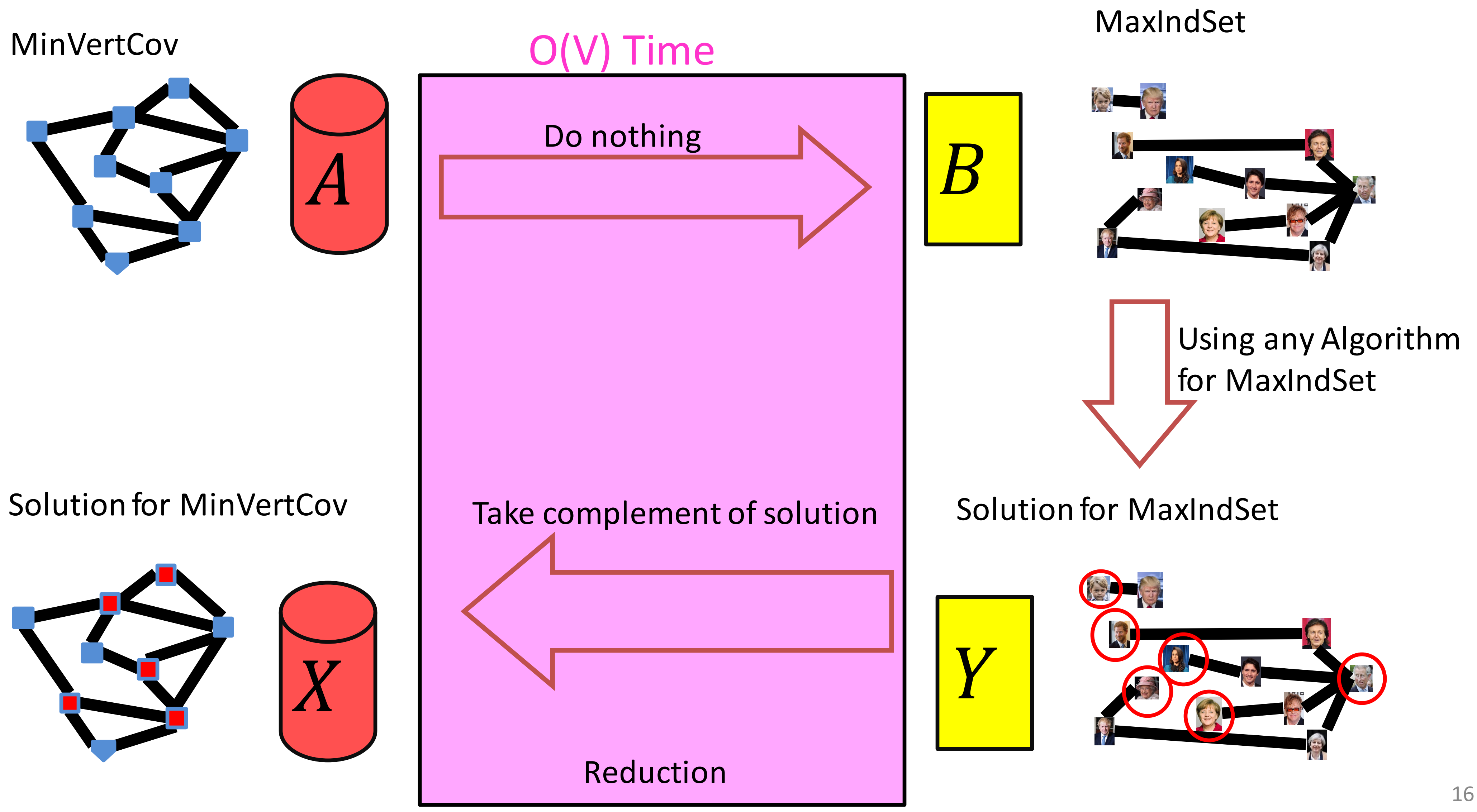
Reduction

Y

Solution for MinVertCov



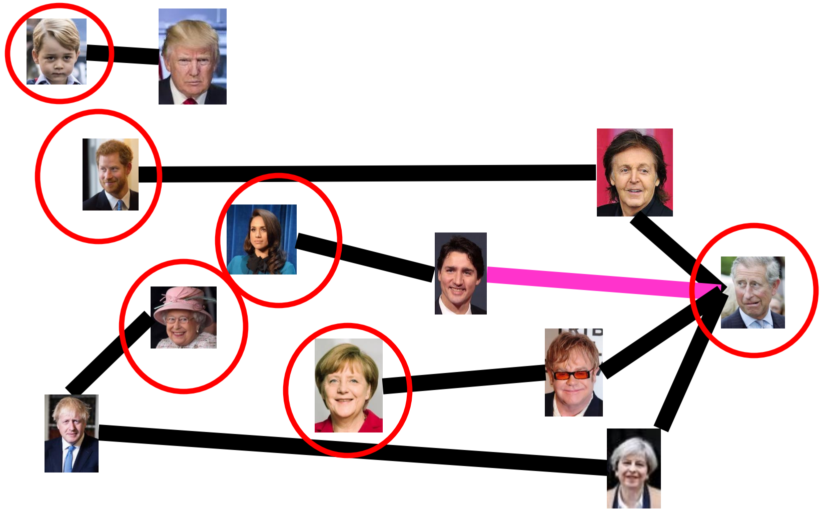
MinVertCover V-Time Reducible to MaxIndSet



Proof: $\Rightarrow$

S is an independent set of G iff $V - S$ is a vertex cover of G

Let S be an independent set



Consider any edge $(x, y) \in E$

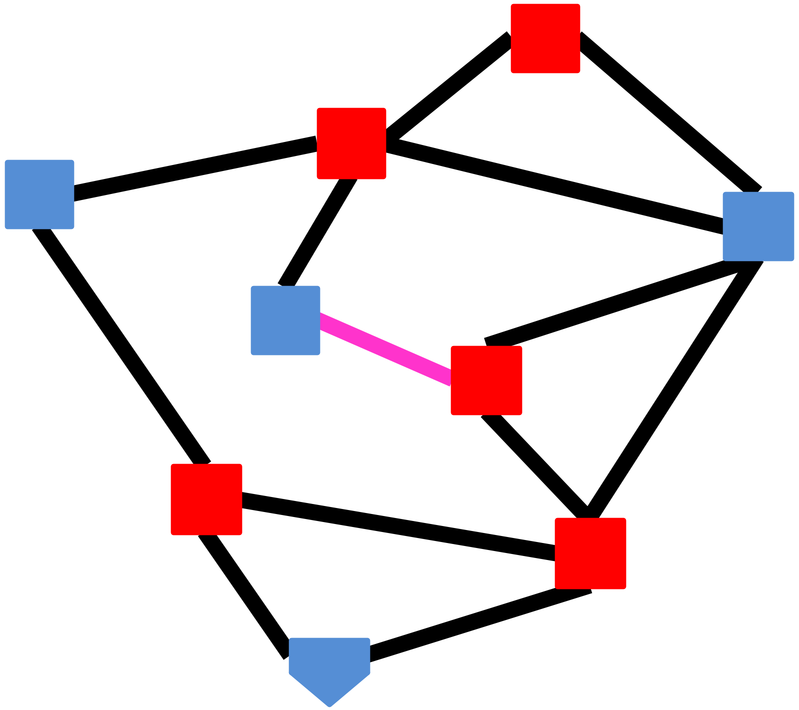
If $x \in S$ then $y \notin S$, because otherwise S would not be an independent set

Therefore $y \in V - S$, so edge (x, y) is covered by $V - S$

Proof: $\Leftarrow$

S is an independent set of G iff $V - S$ is a vertex cover of G

Let $V - S$ be a vertex cover



Consider any edge $(x, y) \in E$

At least one of x and y belong to $V - S$, because $V - S$ is a vertex cover

Therefore x and y are not both in S,

No edge has both end-nodes in S, thus S is an independent set

Conclusion

- MaxIndSet and MinVertCov are either both fast, or both slow
  - Spoiler alert: We don’t know which!
    - (But we think they’re both slow)
  - Both problems are NP-Complete

Why Study NP-Completeness

- All semester, we've studied ***finding algorithms*** to solve problems using various tools.
- Sometimes we instead need to prove that a problem is ***extremely hard***, so as not to waste time on it!
  - NP-Complete Problems are hard
  - Let's go over a few of them quickly
  - Let's show how to prove a new problem is NP-Complete

Some Preliminaries

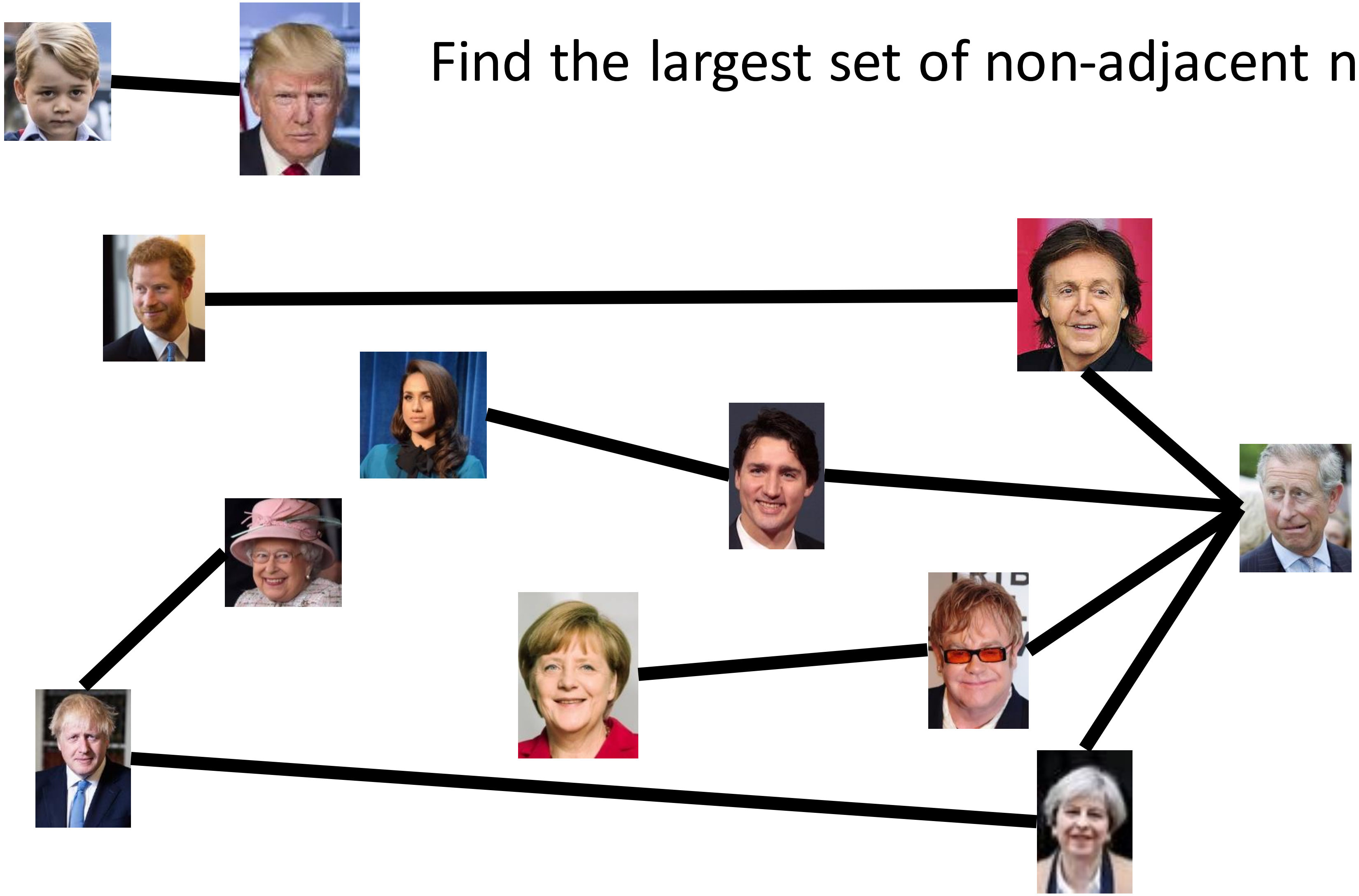
Before we go further on this topic….

- This is a complex (and interesting!) topic in CS theory
- In our few lectures, we may approach things from a simpler viewpoint than you'd get in a CS theory course

- The math and theory related to NP-complete problems starts with ***decision problems***
  - What's that? Let's use independent set and vertex cover as examples
  - What's described next applies to any optimization problems we've seen

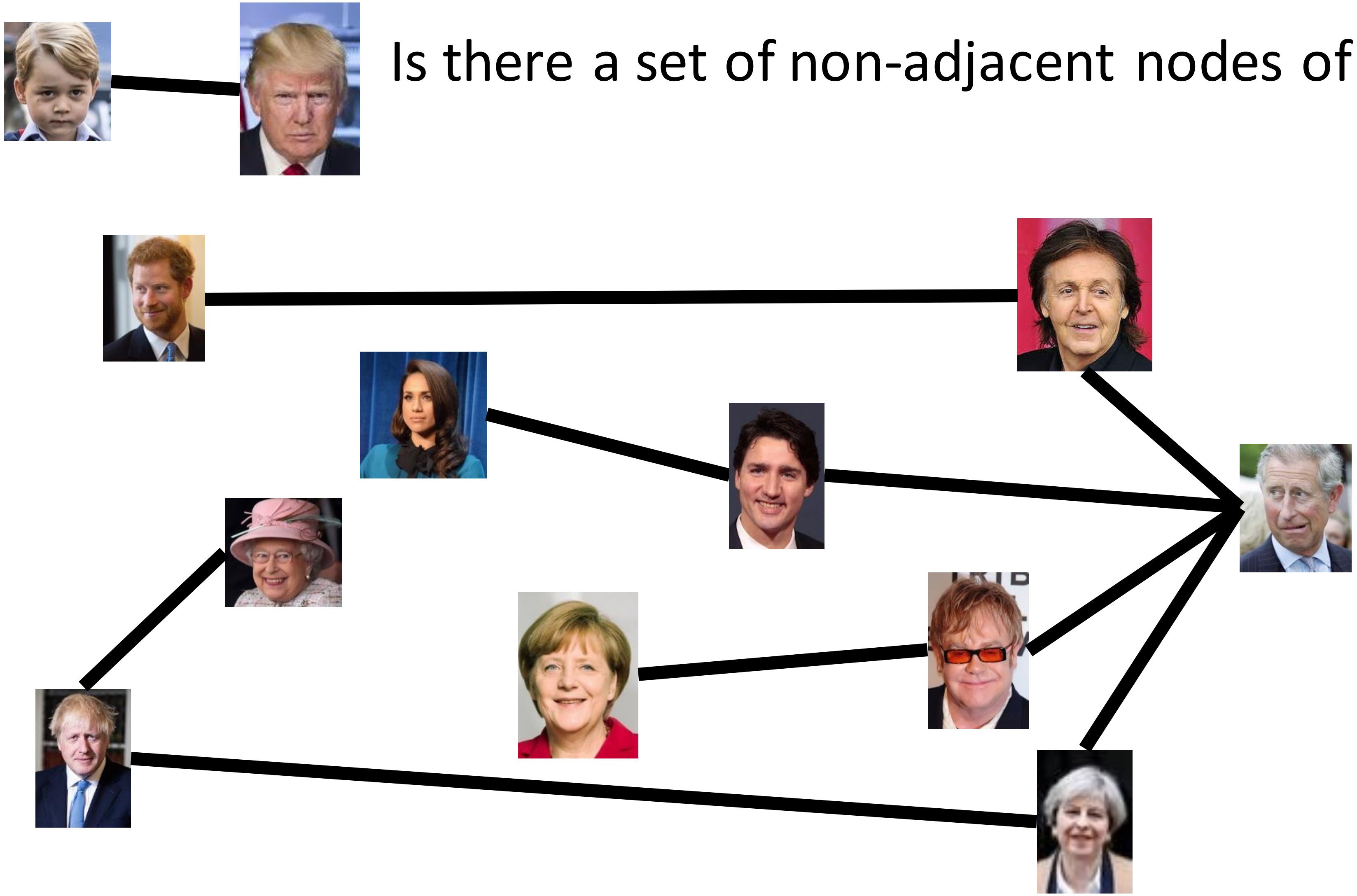
Max Independent Set

Find the largest set of non-adjacent nodes



k Independent Set

Is there a set of non-adjacent nodes of size k?



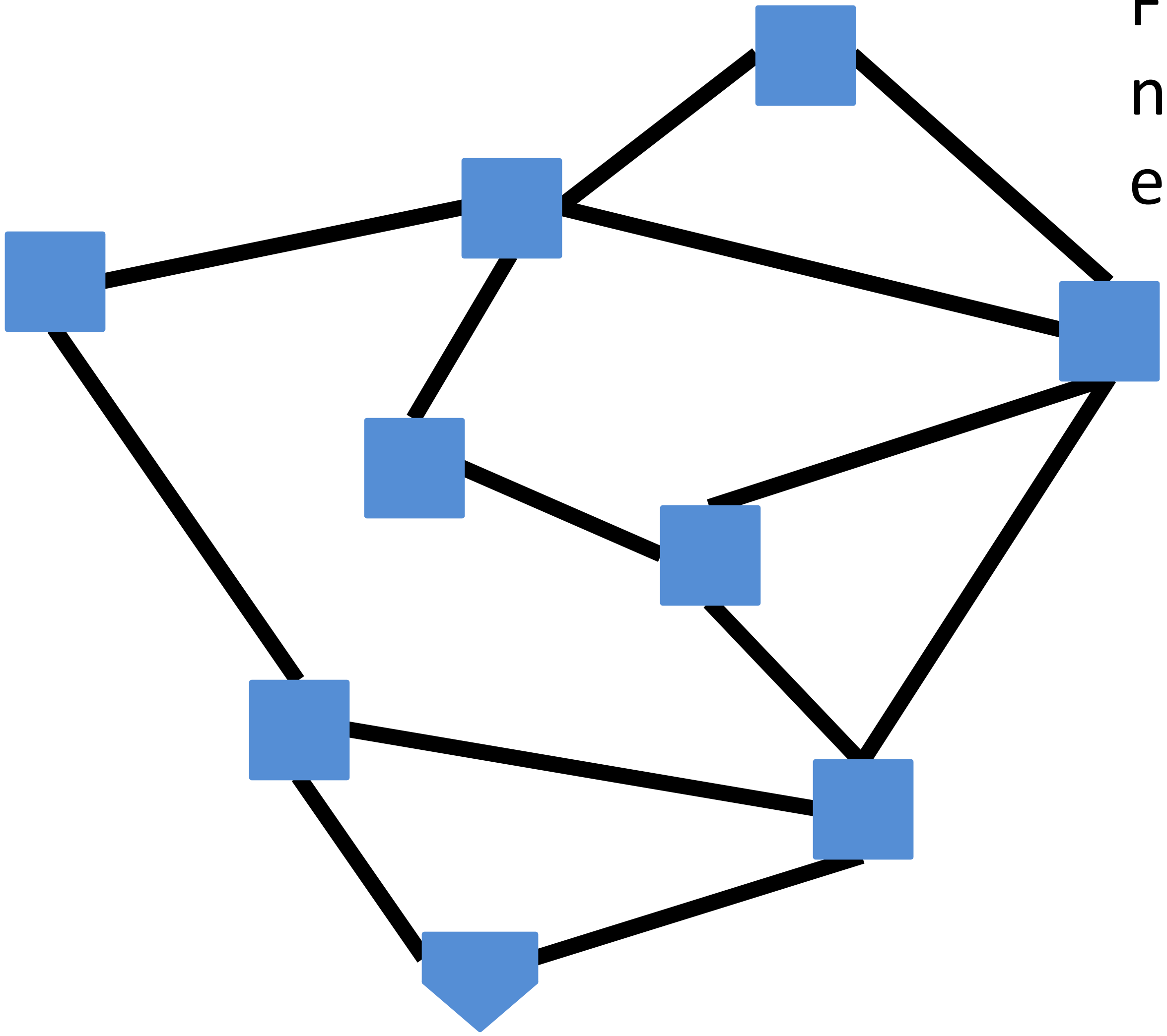
Maximum Independent Set

- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Maximum Independent Set Problem: Given a graph $G = (V, E)$ find the maximum independent set S

k Independent Set

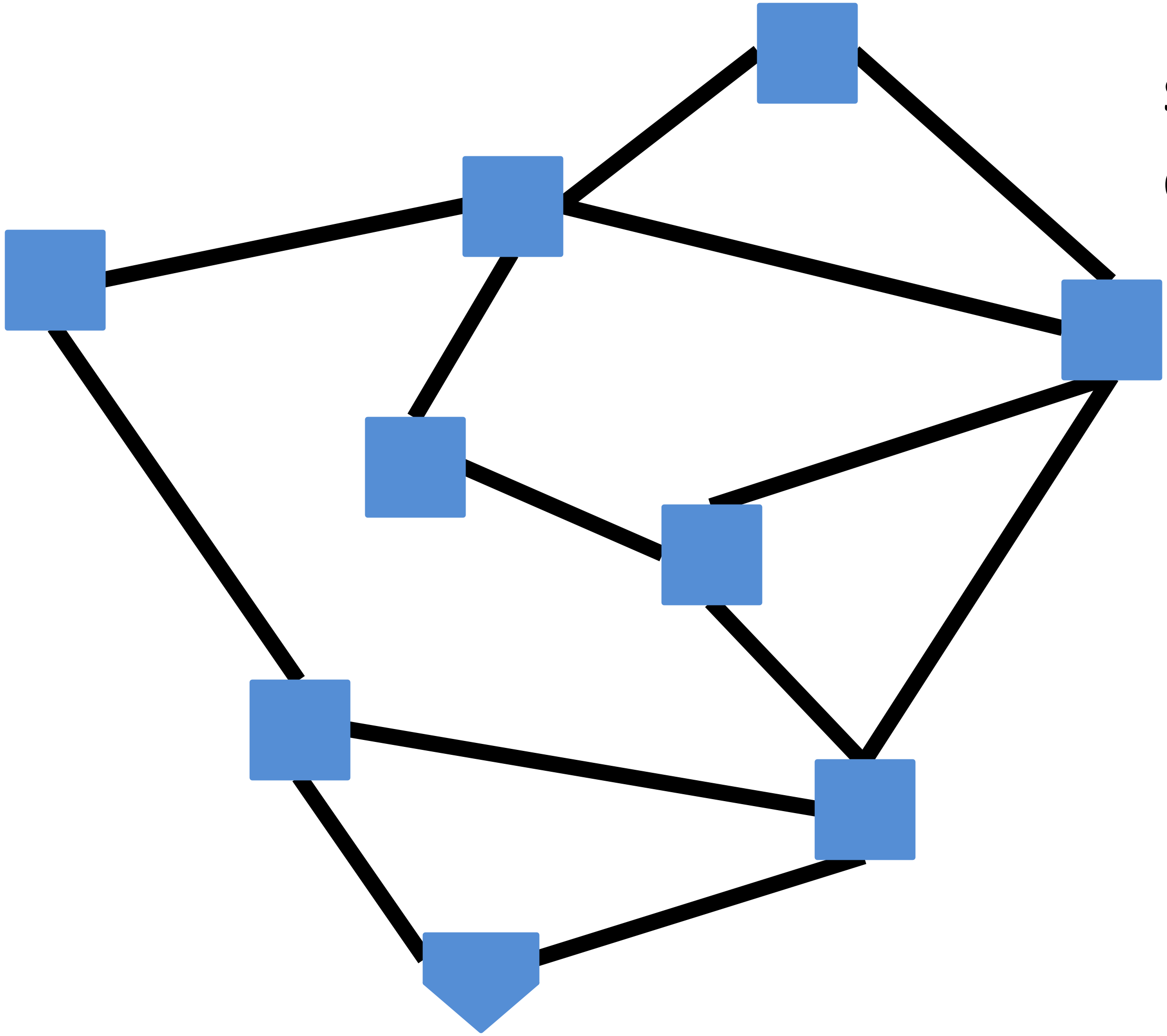
- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- k Independent Set Problem: Given a graph $G = (V, E)$ and a number k, **determine whether there is an independent set $\boldsymbol{S}$ of size $\boldsymbol{k}$**

Min Vertex Cover



Find the smallest set of nodes which covers every edge

k Vertex Cover



Is there a set of nodes of size k which covers every edge?

Minimum Vertex Cover

- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Minimum Vertex Cover: Given a graph $G = (V, E)$ find the minimum vertex cover C

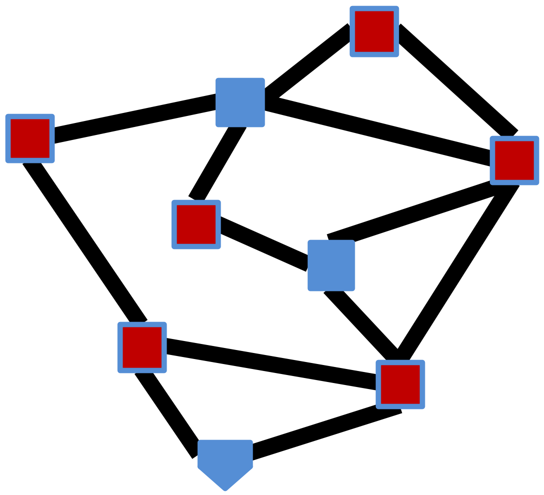
k Vertex Cover

- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- k Vertex Cover: Given a graph $G = (V, E)$ and a number k, **determine if there is a vertex cover $\boldsymbol{C}$ of size $\boldsymbol{k}$**

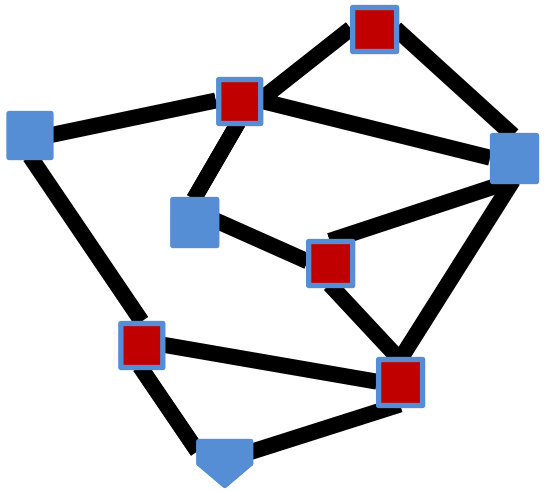
k Vertex Cover

- ***k* Vertex Cover Problem:** Given a graph $G = (V, E)$ <u>and</u> an integer k, **determine if there is a vertex cover C of size k**

True for k=6



True for k=5



Is 5 the smallest?

True for k=4?

Problem Types

- **Decision Problems:**
  - Is there a solution?
    - Result is True/False
  - E.g. Is there a vertex cover of size k?
- **Optimal Value Problems:**
  - E.g. What's the min *k* for *k*-vertex cover decision problem?
- **Search Problems:**
  - Find a solution
    - Result more complex than T/F or a k
  - E.g. Find a vertex cover of size k
- **Verification Problems:**
  - Given a potential solution for an input, is that input valid?
    - Result is True/False
    - For *decision problem*, check solution to its *search problem*
  - E.g. Is **set of vertices** a vertex cover of size k**?**

If we can solve this…

…Then we can solve this,…

…and also this

Looking ahead:
We'll use this to define a problem classes P and NP

Looking ahead:
We'll use this to define a problem class called NP

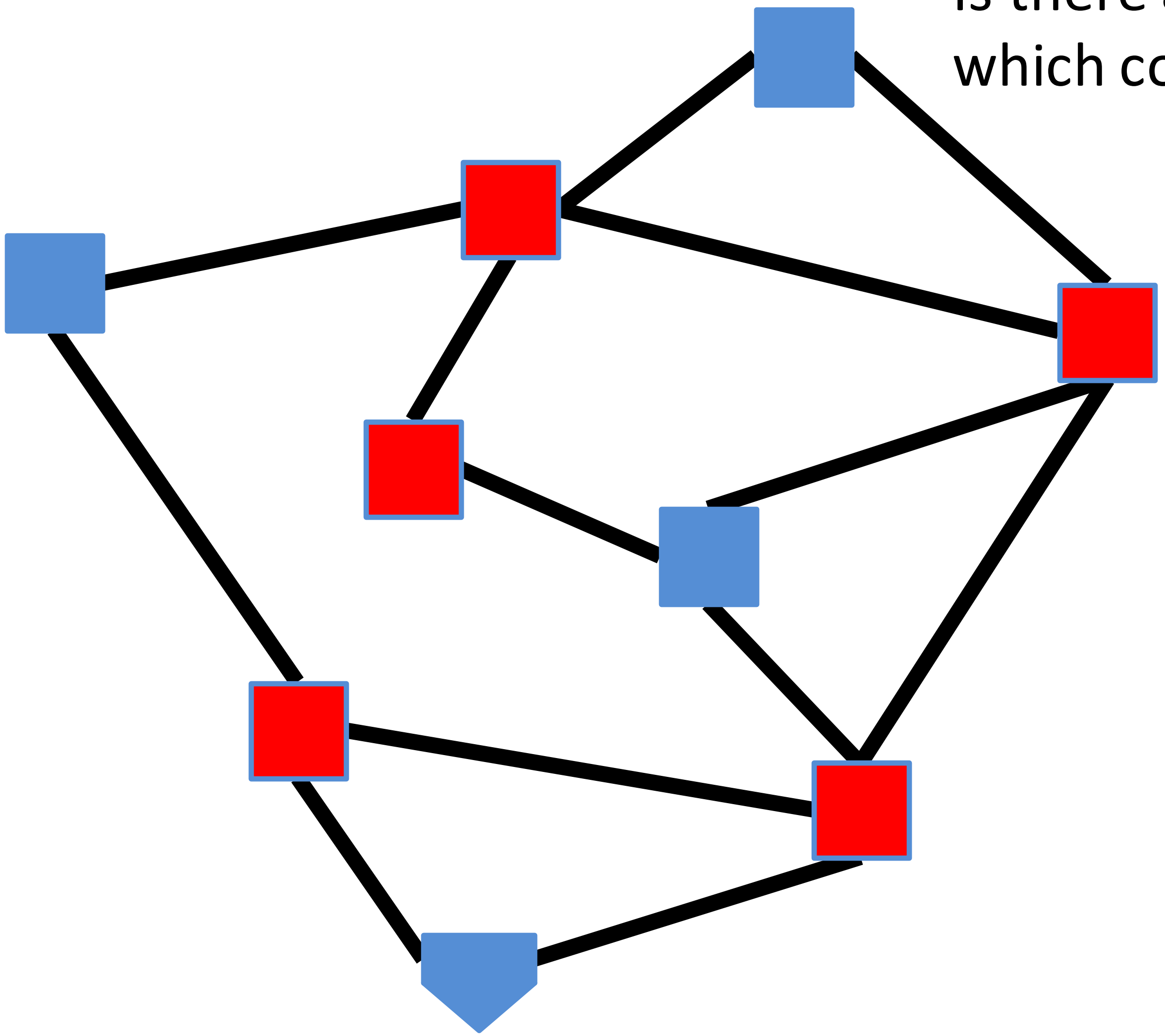
Using a k-VertexCover decider to build a searcher

Note this is a reduction!
kVC-search $\leq_p$ kVC-decider

- Set $i = k - 1$
- Remove nodes (and incident edges) one at a time
- Check if there is a vertex cover of size i (i.e. use the "decider")
  - If so, then that removed node was part of the k vertex cover, set $i = i - 1$
  - Else, it wasn't

Did I need this node to cover its edges to have a vertex cover of size k?

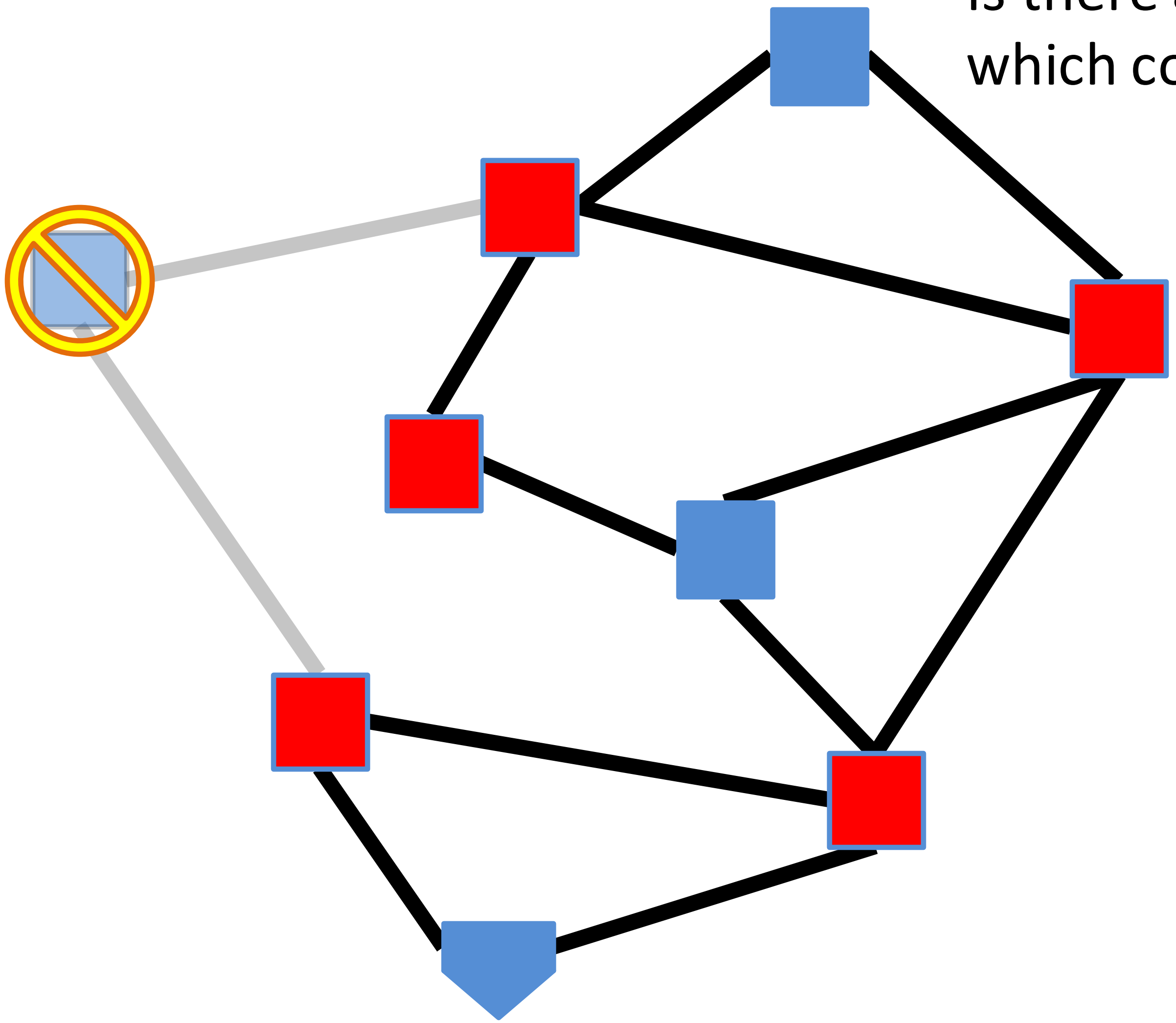
5 Vertex Cover (Decision)



Is there a set of nodes of size 5 which covers every edge?

Yes!

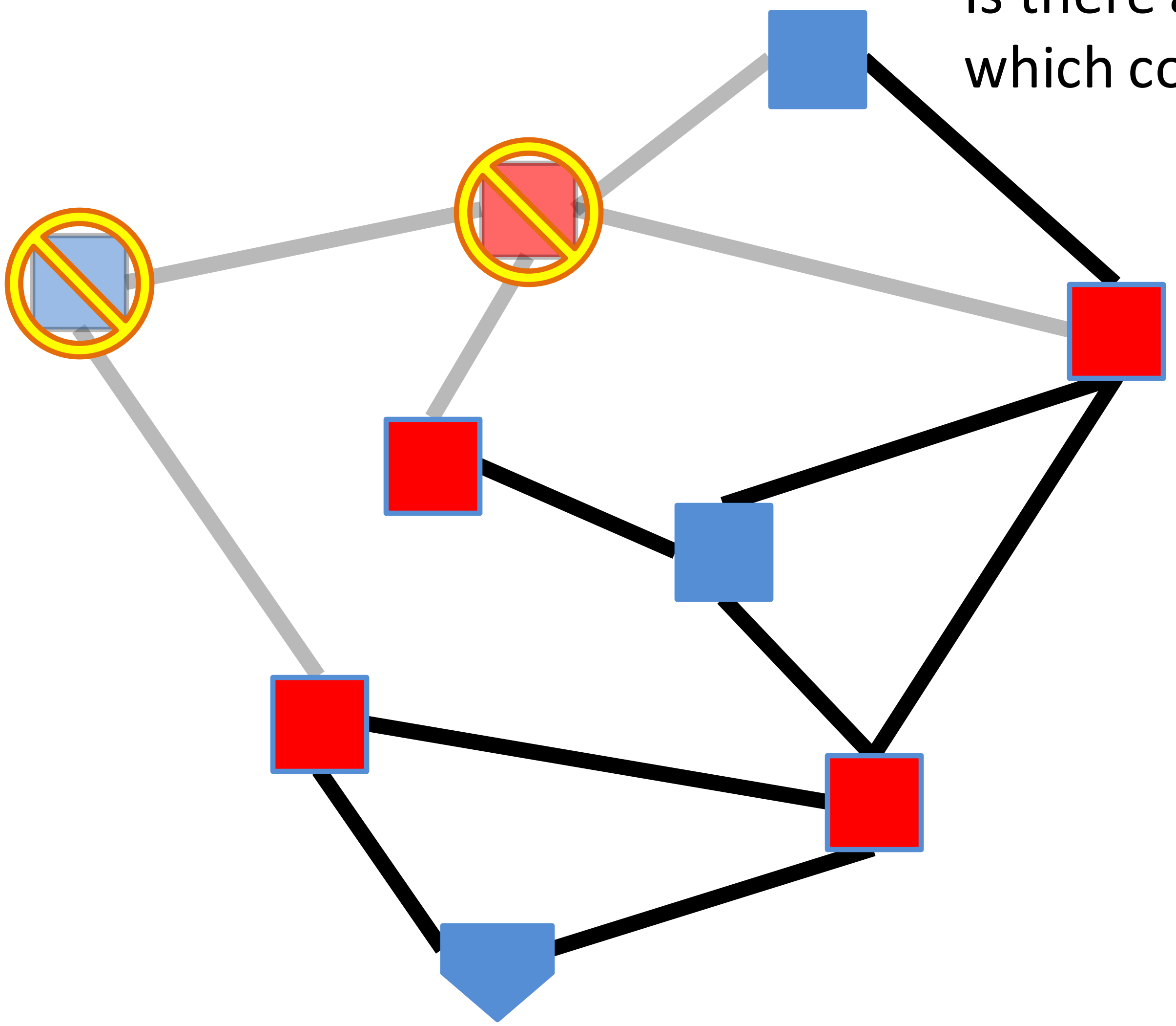
4 Vertex Cover (Decision)



Is there a set of nodes of size 4 which covers every edge?

No!

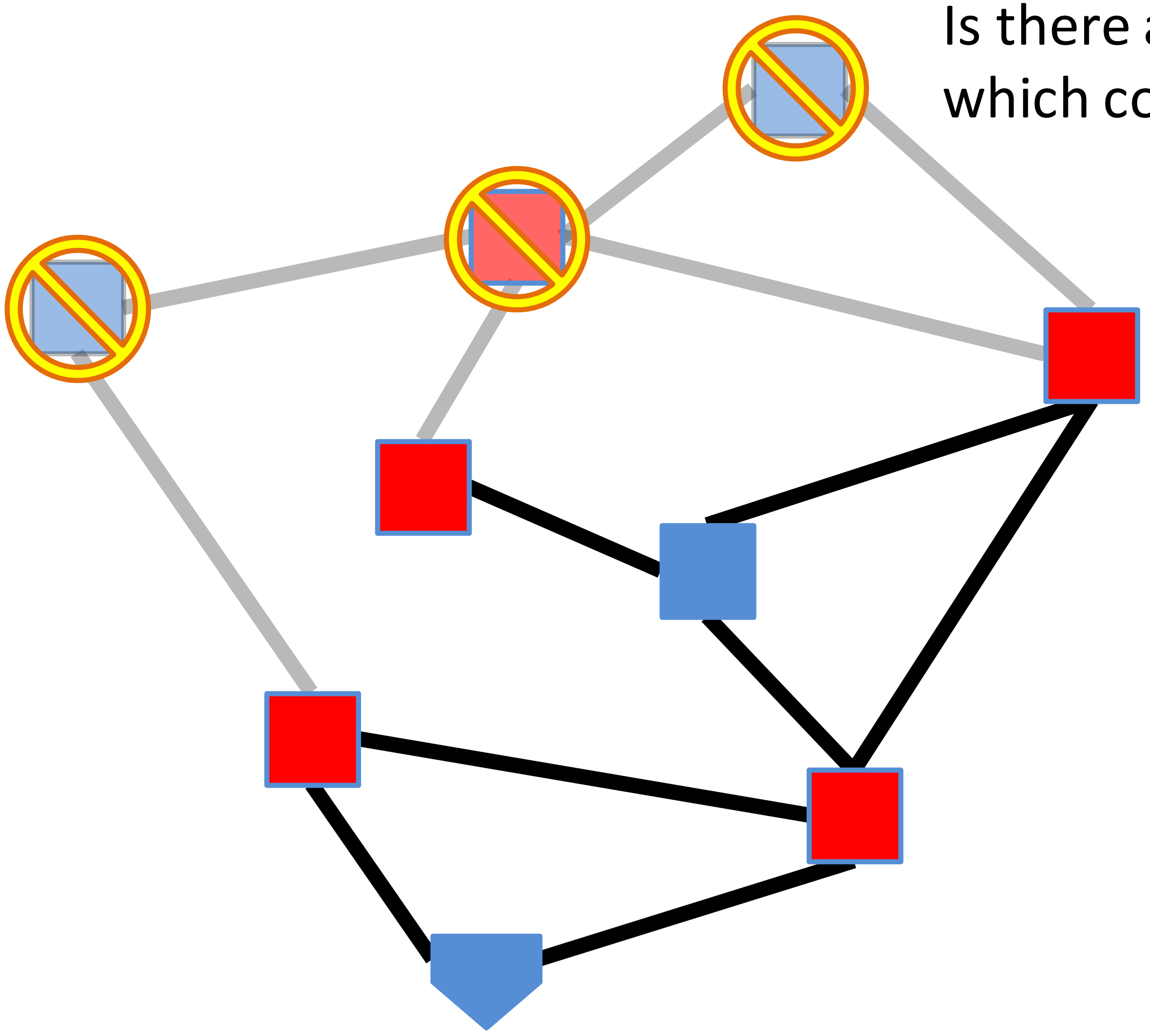
4 Vertex Cover (Decision)



Is there a set of nodes of size 4 which covers every edge?

Yes!

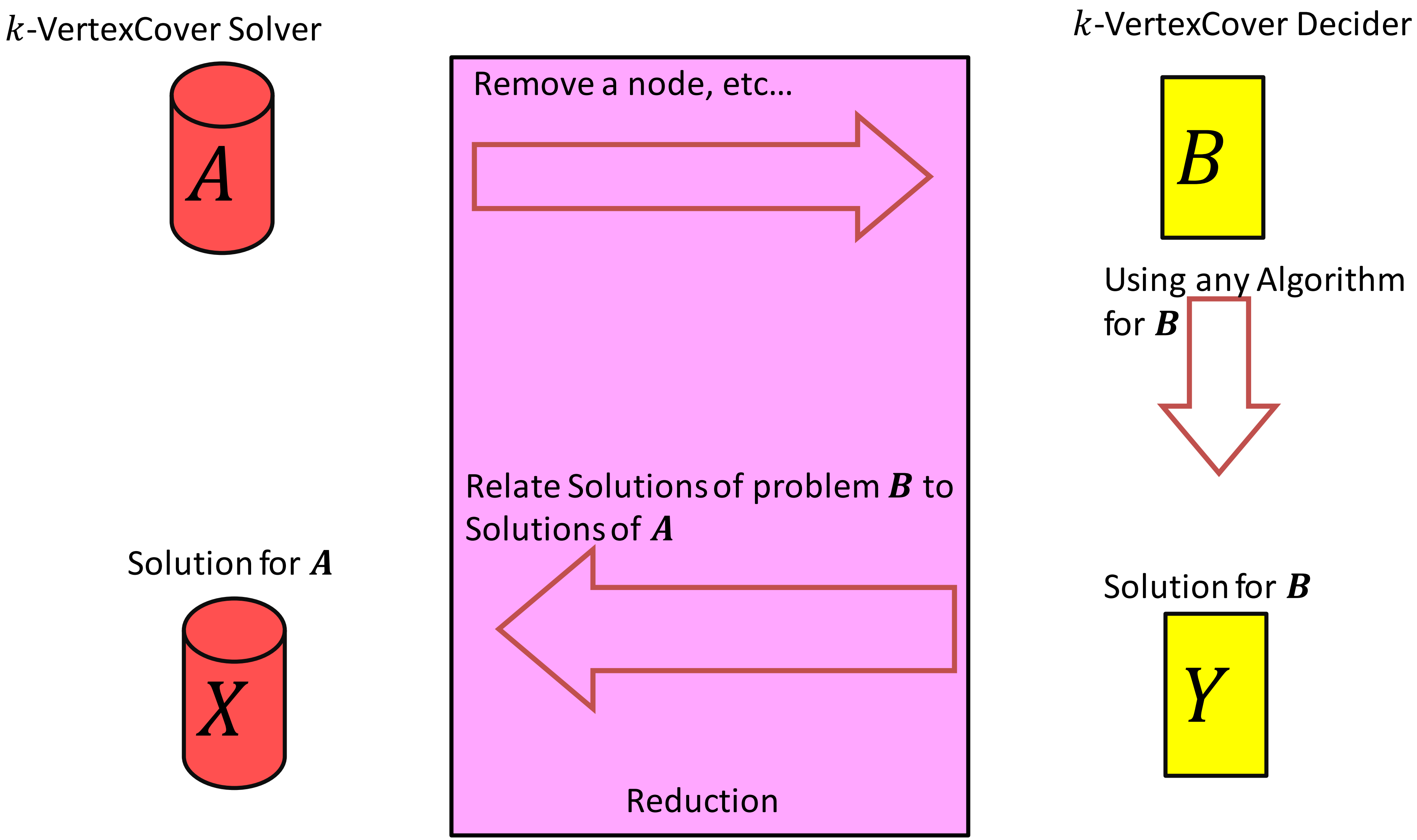
3 Vertex Cover (Decision)



Is there a set of nodes of size 3 which covers every edge?

No!

Reduction


k-VertexCover Solver
A
Remove a node, etc…
k-VertexCover Decider
B
Using any Algorithm for $\boldsymbol{B}$
Relate Solutions of problem $\boldsymbol{B}$ to Solutions of $\boldsymbol{A}$
Solution for $\boldsymbol{A}$
X
Solution for $\boldsymbol{B}$
Y
Reduction

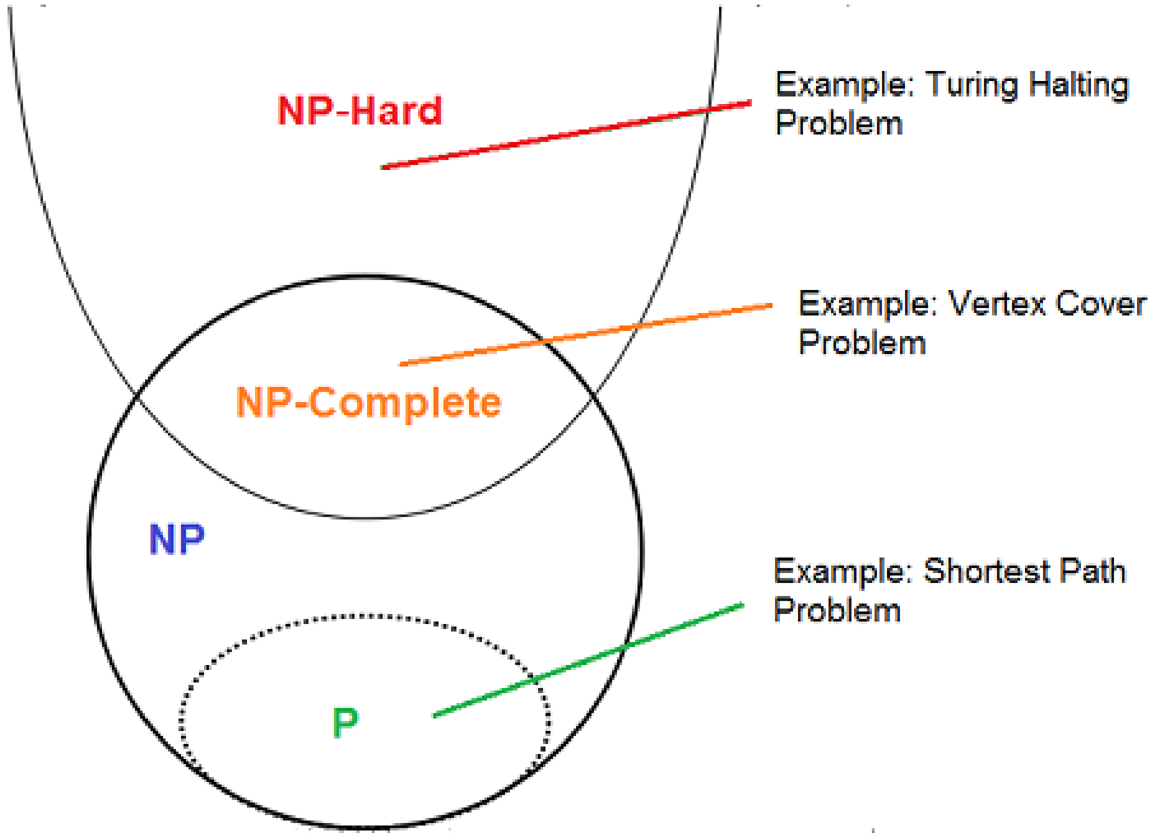
Quick Background!

- ***P***: Set of problems solved in polynomial time (e.g., sorting a list)

- ***NP***: Set of problems that can be:
  1) Solved in non-deterministic polynomial time
  2) A solution verified in polynomial time

- ***NP-Hard***: Set of problems that are as hard as (or harder) than the hardest problems in NP

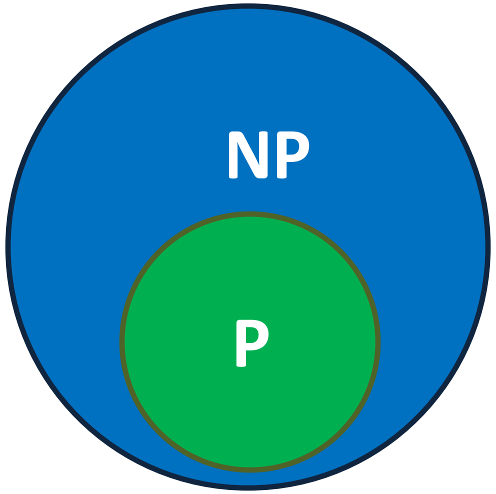
- ***NP-Complete***: Set of problems that are both NP and NP-Hard (i.e., the equally hardest problems in NP)



This diagram assumes that P != NP

Classes of Problems: P vs NP

- P
  - Deterministic Polynomial Time
  - P is the set of problems solvable in polynomial time
    - $O(n^c)$ for some number c
- NP
  - Non-Deterministic Polynomial Time
  - NP is the set of problems ***verifiable*** in polynomial time
    - Verify a proposed solution (not find one) in $O(n^c)$ for some number c
    - For decision problems, really verifying using some information we call a *certificate*
- Open Problem: Does P=NP?
  - Certainly $P \subseteq NP$



k-Independent Set is NP

- To show: Given a potential solution, can we **verify** it in $O(n^p)$? $[n = V + E]$

How can we verify it?

1. Check that it's of size k $O(V)$
2. Check that it's an independent set $O(V^2)$

k-Vertex Cover is NP

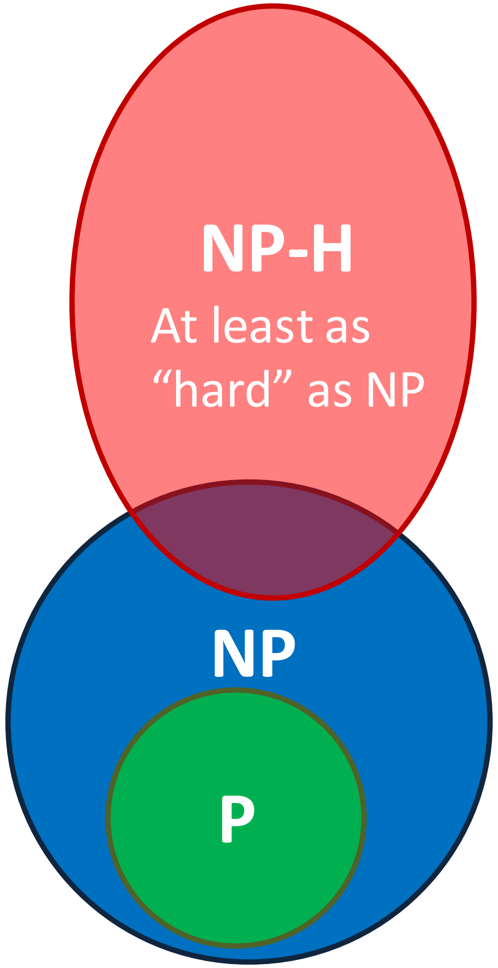
- To show: Given a potential solution, can we **verify** it in $O(n^p)$? $[n = V + E]$

How can we verify it?

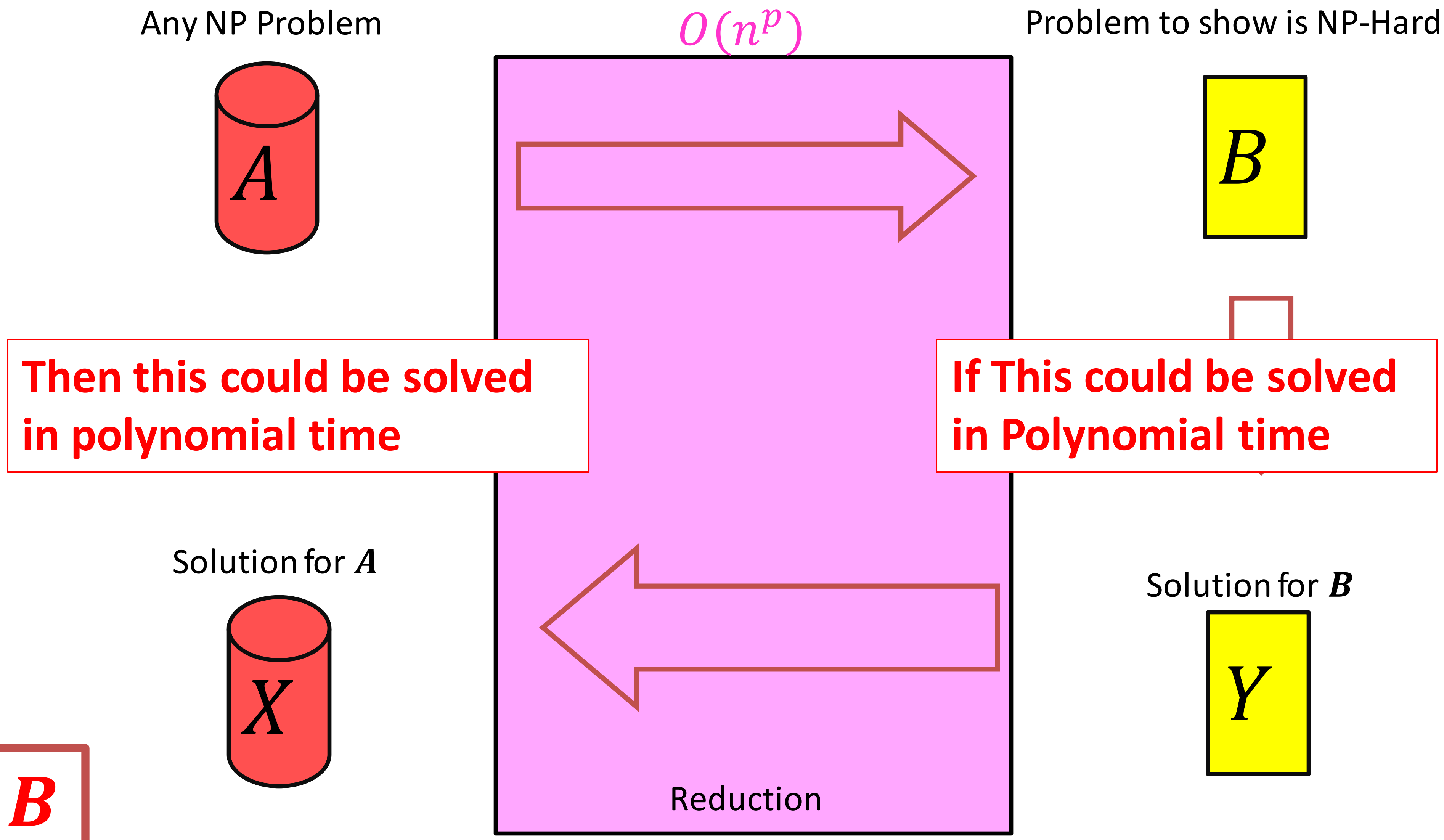
1. Check that it's of size k $O(V)$
2. Check that it's a Vertex Cover $O(E)$

NP-Hard

- How can we try to figure out if P=NP?
- Identify problems at least as "hard" as NP
  - If any of these "hard" problems can be solved in polynomial time, then all NP problems can be solved in polynomial time.
- Definition: NP-Hard:
  - B is NP-Hard if $\forall A \in NP, A \leq_p B$
  - $A \leq_p B$ means A reduces to B in polynomial time
  - Remember: $A \leq_p B$ implies A is not harder than B



NP-Hardness Reduction



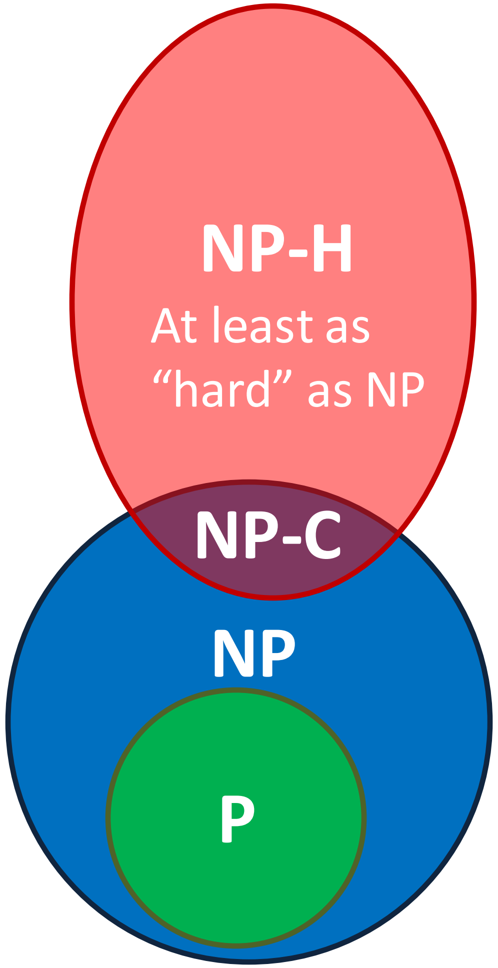
$\boldsymbol{A \leq_p B}$

NP-Complete

NP-Complete = NP ∩ NP-Hard

- The "hardest" of all the problems in NP
- An NP-C problem is polynomial iff all NP problems are polynomial. I.e. P=NP
- If P=NP, then all NP-C problems are polynomial
- "Together they stand, together they fall"

- **How to show a problem C is NP-Complete?**
  - Show C belongs to NP
    - Show we can verify a solution in polynomial time
  - Show C is NP-Hard
    - $\forall A \in NP, A \leq_p C$ (That sounds really hard to do!)
    - Or, show a reduction from another NP-Hard problem. (Why? Details next.)



We now just need a FIRST NP-Hard problem

NP-Completeness

- So…a problem is NP-Complete if you can do the following:

- 1) Show how to verify it in polynomial time
  - Given a solution to the problem, verify it is correct
  - That algorithm's runtime needs to be a polynomial (usually easy)

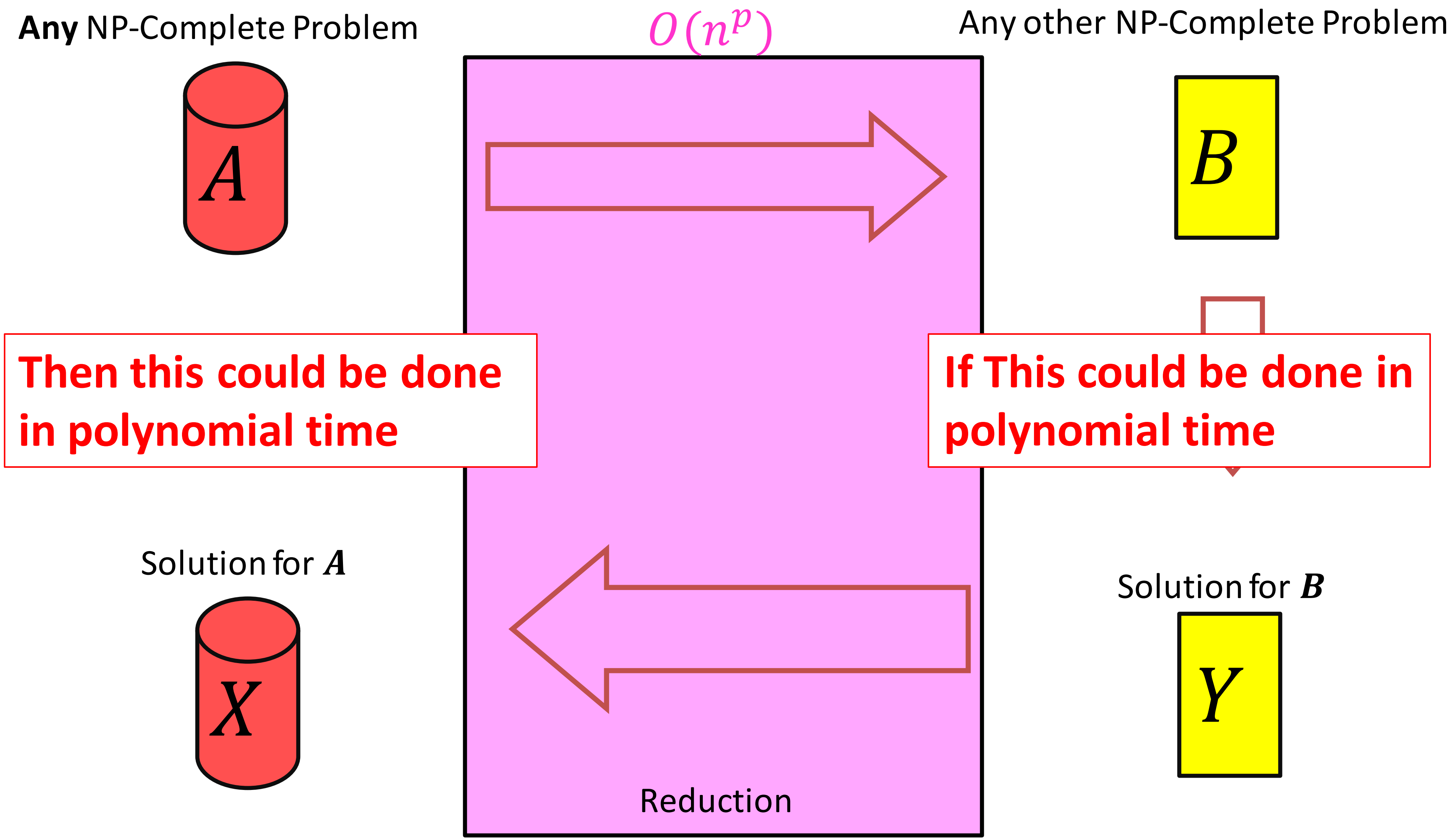
- 2) Show the problem is NP-Hard (as hard or harder than a known NP-Hard Problem)
  - Take a currently known NP-Hard problem (let's call it A)
  - Show that $A \leq_p X$ (where X is your problem)
  - Why? If A is NP-Hard, then: *any NP problem* $\leq_p A$
  - Transitivity: *any NP problem* $\leq_p A \leq_p X$
  - So X satisfies definition of NP-Hard

“Consequences” of NP-Completeness

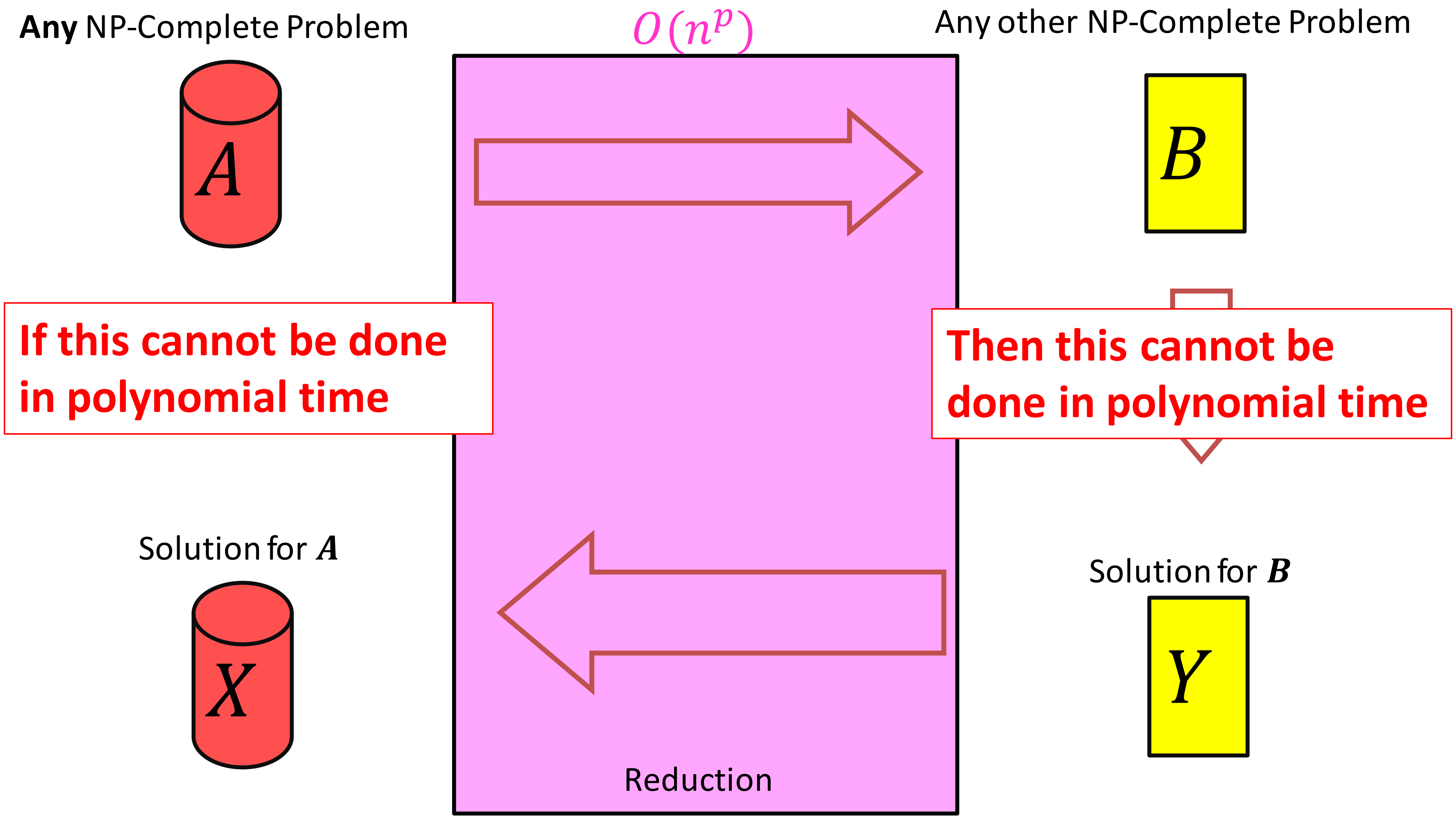
- NP-Complete is the set of “hardest” problems in NP, with these important properties:
  - If any *one* NP-Complete problem can be solved in polynomial time…
  - …then *every* NP-Complete problem can be solved in polynomial time…
  - …and in fact *every* problem in **NP** can be solved in polynomial time (which would show **P = NP**)
  - Or, prove an exponential lower-bound for *any single* **NP-hard** problem, then *every* **NP-hard** problem (including **NP-C**) is exponential

Therefore: solve (say) traveling salesperson problem in O(n^{100}) time, you've proved that **P = NP**. Retire rich & famous!

NP-Completeness: $A \leq_p B$ and B in P



NP-C: $A \le_p B$ and we prove A <u>not</u> in P



Summary of Where We Are

- Focusing on “hard” problems, those that seem to be exponential
- Reductions used to show “hardness” relationships between problems
- Starting to define “classes” of problems based on complexity issues
  - **P** are problems that can be solved in polynomial time
  - **NP** are problems where a solution can be verified in polynomial time
  - **NP-hard** are problems that are at least as hard as anything in NP
  - **NP-complete** are NP-hard problems that “stand or fall together”

Review: P and NP Summary

- **P** = set of problems that can be solved in polynomial time
- **NP** = set of problems for which a solution can be verified in polynomial time
  - Note: this is a more “informal” definition, but it’s fine for CS4102
  - See later slide on ”certificates” for more info.
- **P** $\subseteq$ **NP**
- Open question: Does **P = NP**?

More Reminders and Some Consequences

- **Definition of NP-Hard and NP-Complete:**
  - If all problems A $\in$ **NP** are reducible to B , then B is *NP-Hard*
  - We say B is *NP-Complete* if:
    - B is NP-Hard
    - and B $\in$ **NP**

- Any NP-C must reduce to any other NP-C. Can you see why?

- If B $\leq_p$ C and B is NP-Complete, C is also NP-Complete
  - Don’t see why? We’ll show details in two more slides
  - As long as C $\in$ **NP**. Otherwise can only say C $\in$ **NP-hard.**

3-SAT

- Shown to be NP-Hard by Cook and Levin (independently)
- Given a 3-CNF formula (logical AND of clauses, each an OR of 3 variables), Is there an assignment of true/false to each variable to make the formula true?

$$\underbrace{(x \vee y \vee z)}_{\text{Clause}} \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

Variables

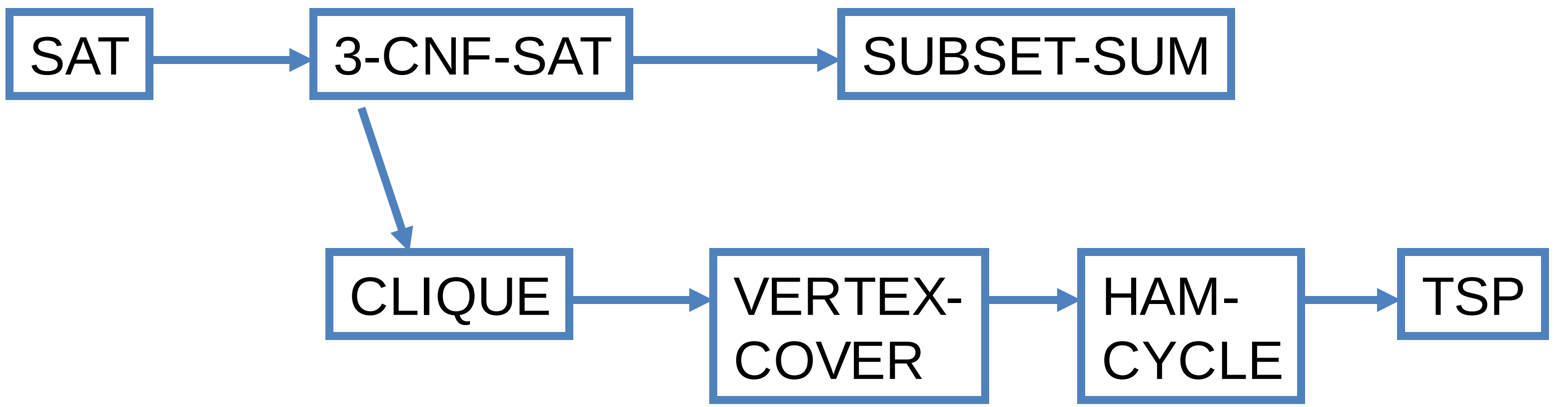
$$\boldsymbol{x = true}$$
$$\boldsymbol{y = false}$$
$$\boldsymbol{z = false}$$
$$\boldsymbol{u = true}$$

Conjunctive Normal Form (CNF)

- Even if the form of the Boolean expression is simplified, the problem may be NP-Complete
  - *Literal*: an occurrence of a Boolean or its negation
  - A Boolean formula is in *conjunctive normal form*, or *CNF*, if it is an AND of clauses, each of which is an OR of literals
    - Ex: $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_5)$
  - *3-CNF*: each clause has exactly 3 distinct literals
    - Ex: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_5 \vee x_3 \vee x_4)$
    - Notice: true if at least one literal in each clause is true
  - Note: Arbitrary SAT expressions can be translated into CNF forms by introducing intermediate variables etc.

Joining the Club

- Given one NP-Complete problem, others can join the club
  - Prove that SAT reduces to another problem, and so on…



  - Membership in NP-Complete grows…
  - Classic textbook: Garey, M. and D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness,* 1979.

k-Independent Set is NP-Complete

1. Show that it belongs to NP
   - Give a polynomial time verifier
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - Show $3SAT \leq_p kIndSet$

Remember: k-Independent Set is NP

- To show: Given a certificate ("solution" for the search problem), can we **verify** it in $O(n^p)$? $[n = V + E]$

  How can we verify it?

  1. Check that it's of size k $O(V)$
  2. Check that it's an independent set $O(V^2)$

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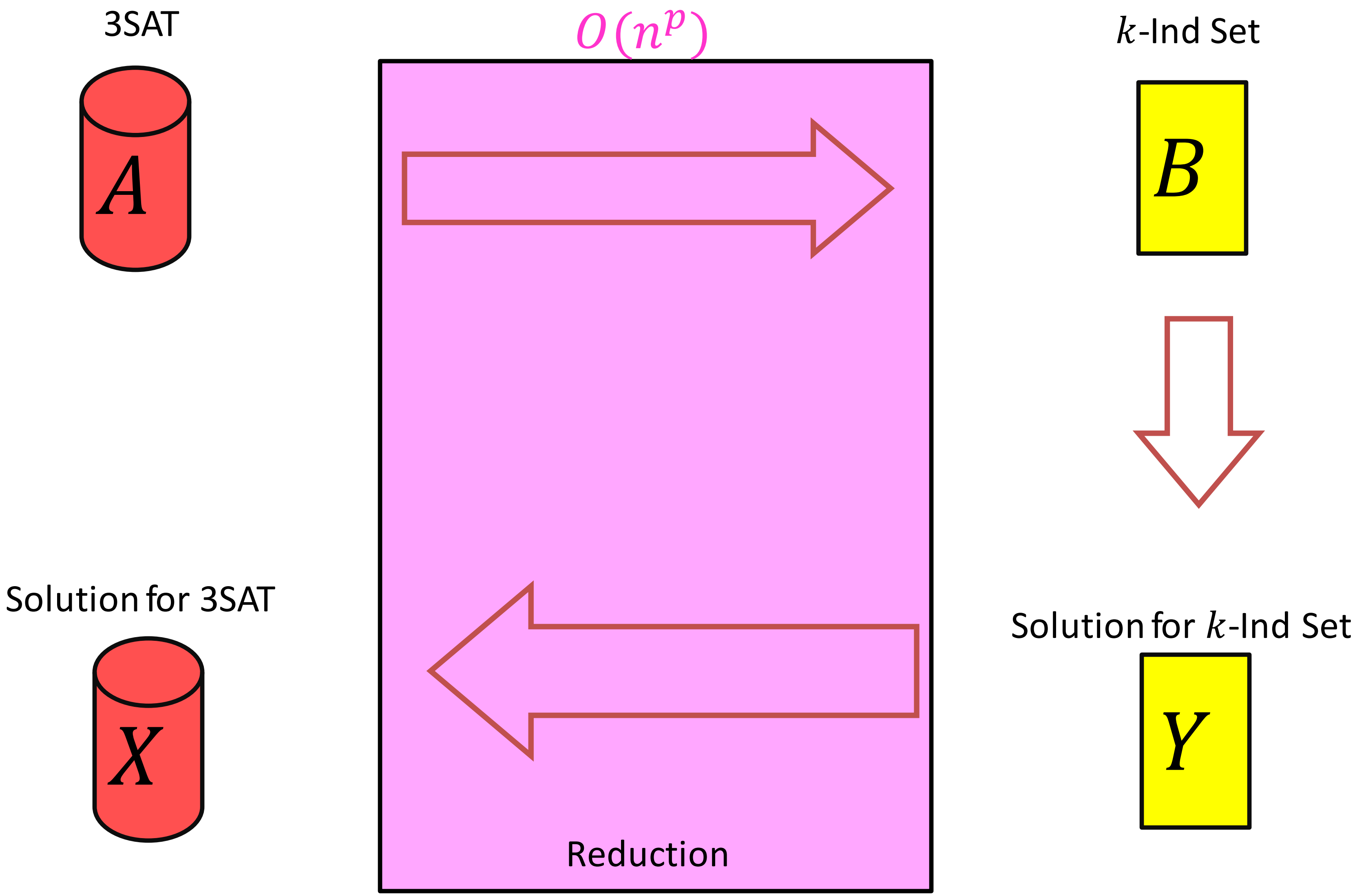
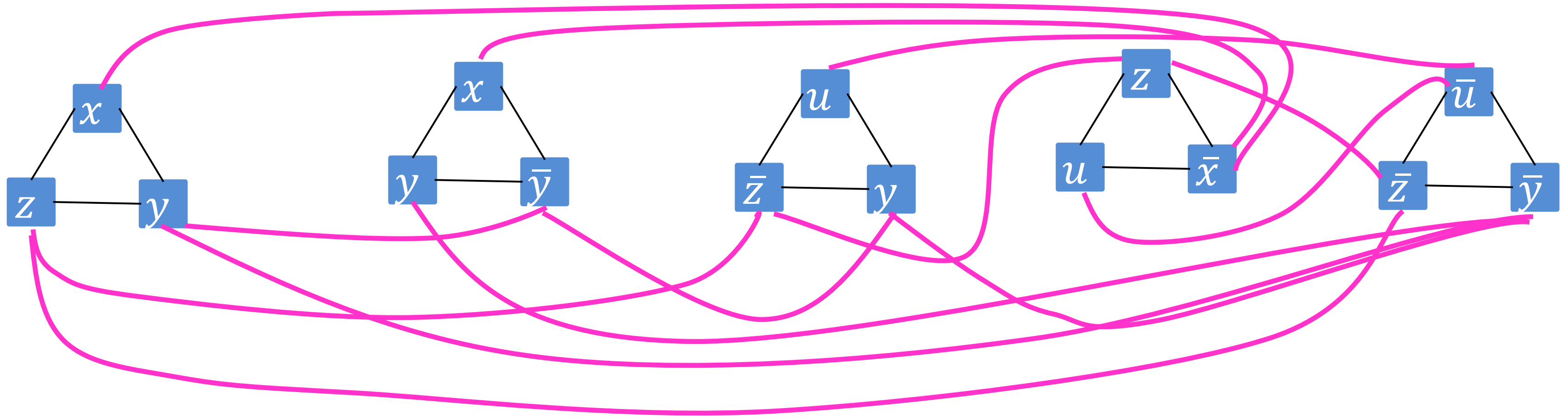
$3SAT \leq_p kIndSet$



Instance of 3SAT to Instance of kIndSet

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



For each clause, produce a triangle graph with its three variables as nodes

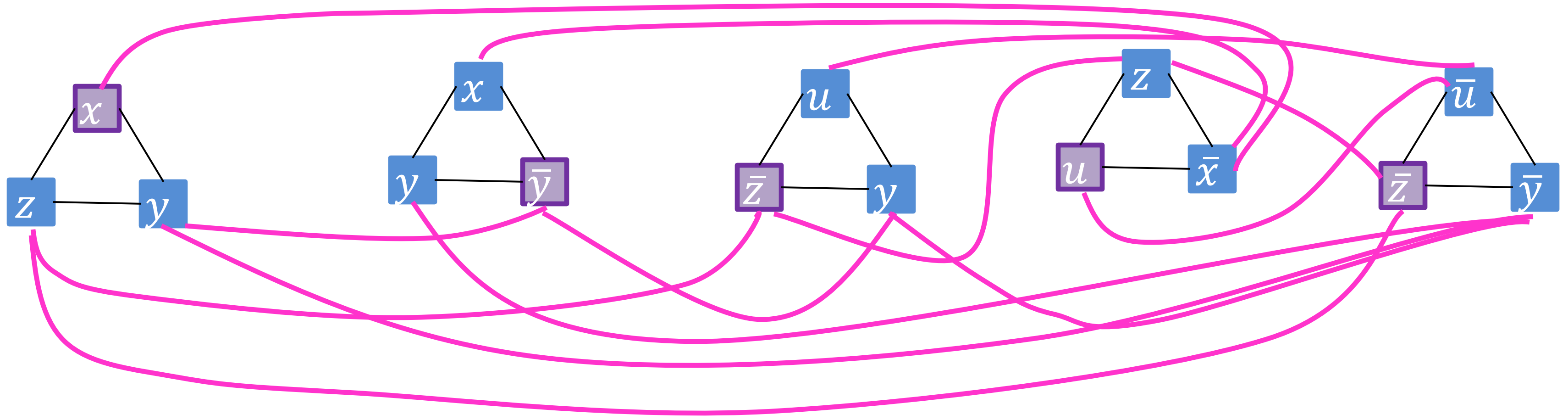
Connect each node to all of its opposites

Let $k =$ number of clauses

There is a k-IndSet in this graph **iff** there is a satisfying assignment

kIndSet ⇒ Satisfying Assignment

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$



$x = true$
$y = false$
$z = false$
$u = true$

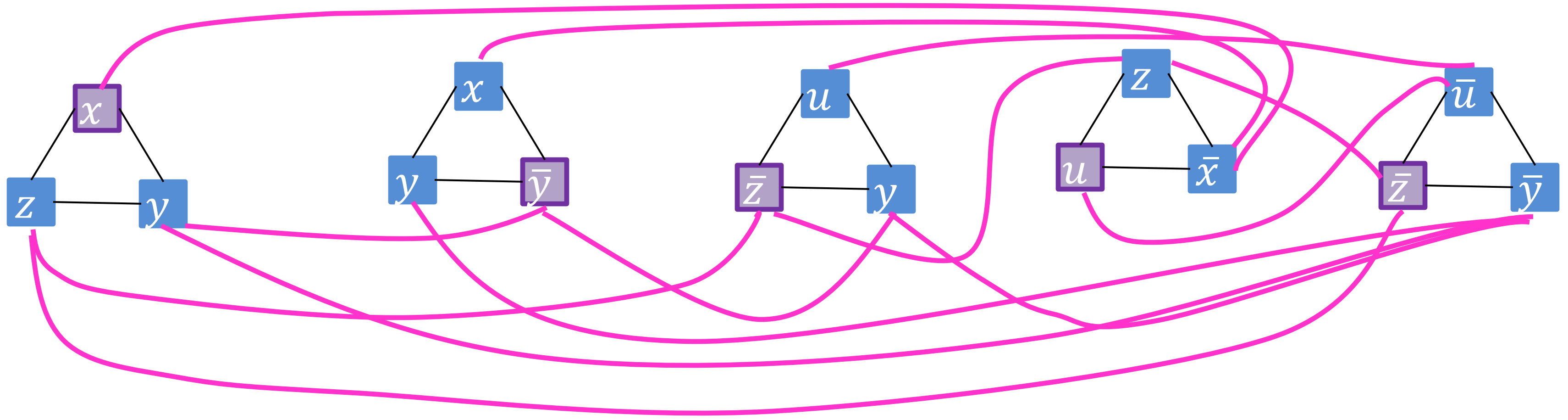
One node per triangle is in the Independent set:
because we can have exactly k total in the set, and 2 in a triangle would be adjacent

If x is selected in some triangle, $\bar{x}$ is not selected in any triangle:
Because every x is adjacent to every $\bar{x}$

Set the variable which each included node represents to "true"

Satisfying Assignment ⇒ kIndSet

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



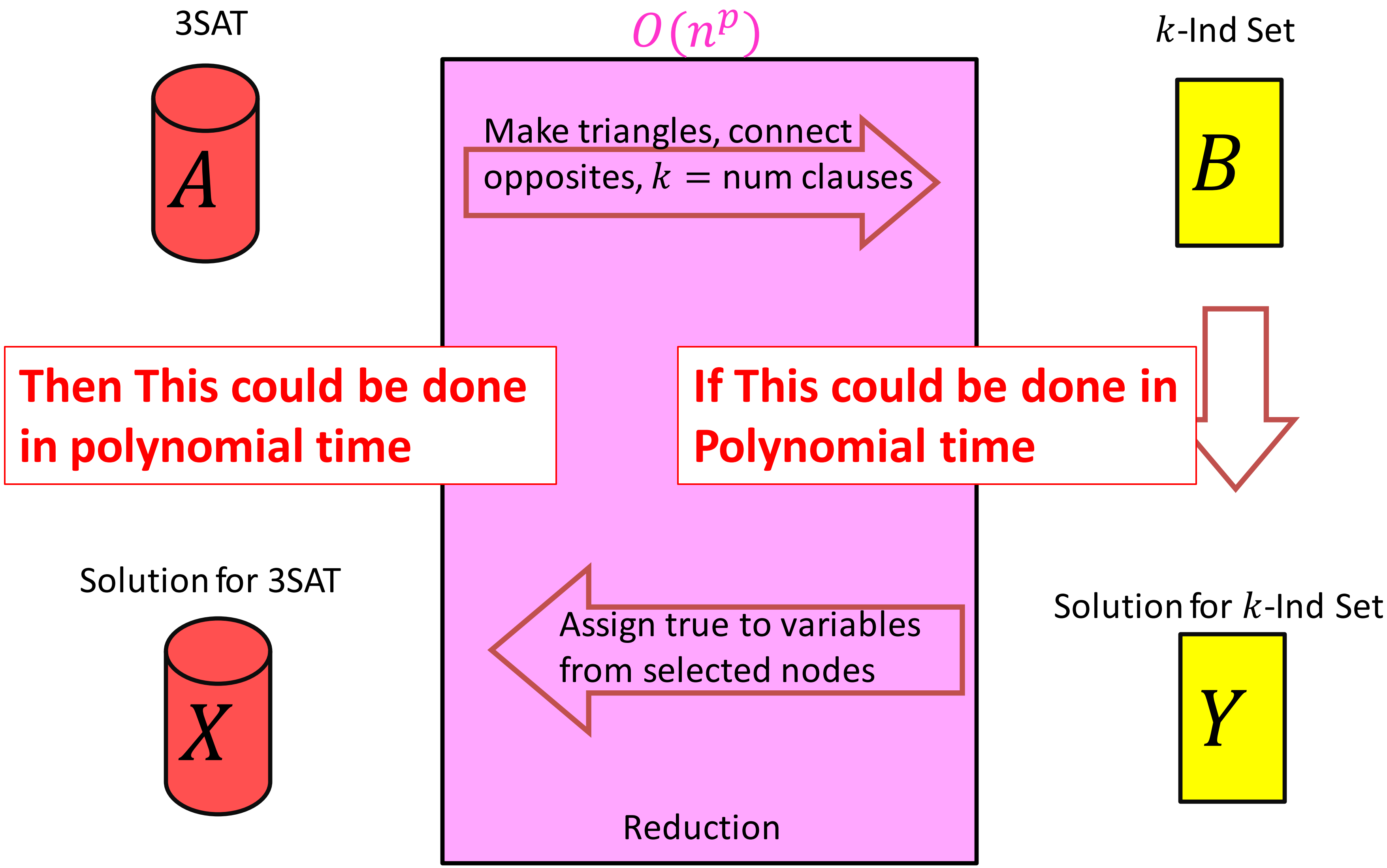
$\boldsymbol{x = true}$
$\boldsymbol{y = false}$
$\boldsymbol{z = false}$
$\boldsymbol{u = true}$

Use one true variable from the assignment for each triangle

The independent set has k nodes, because there are k clauses

If any variable x is true then $\bar{x}$ cannot be true

$3SAT \leq_p kIndSet$



k-Vertex Cover is NP-Complete

1. Show that it belongs to NP
   - Give a polynomial time verifier
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - We showed $kIndSet \leq_p kVertCov$

Remember: k-Vertex Cover is NP

- To show: Given a certificate (“solution” for the search problem), can we verify it in $O(n^p)$? $[n = V + E]$

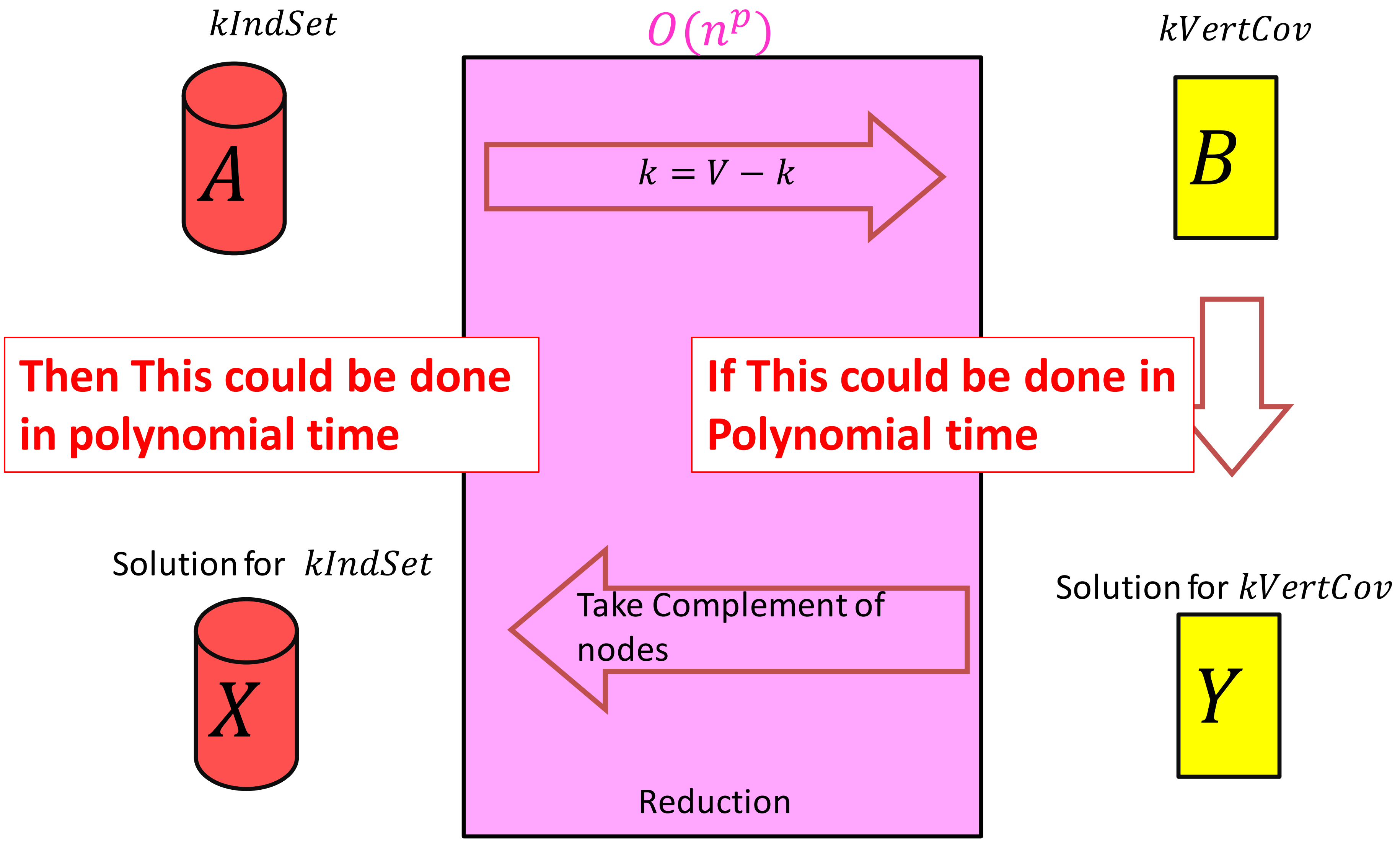
  How can we verify it?

  1. Check that it’s of size k $O(V)$
  2. Check that it’s a Vertex Cover $O(E)$

k-Vertex Cover is NP-Complete

1. Show that it belongs to NP
   - Give a polynomial time verifier
2. Show it is NP-Hard
   - Give a reduction from a known NP-Hard problem
   - We showed $kIndSet \leq_p kVertCov$

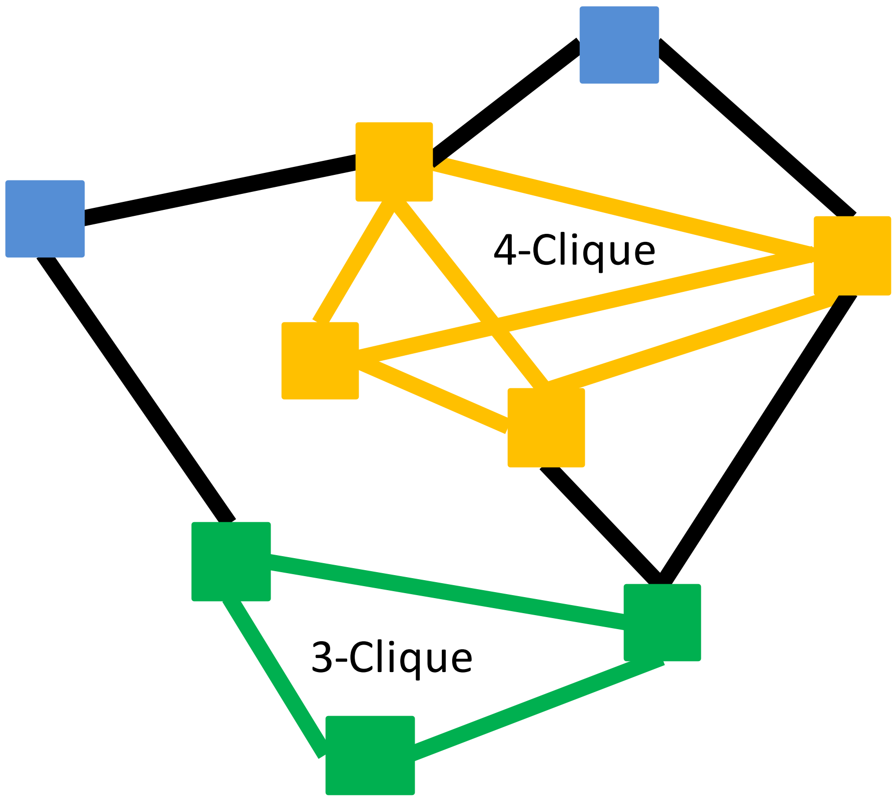
Remember: kIndSet $\leq_p$ kVertCov



k-Clique Problem

Given a graph G and a number k, is there a *clique* of size k?

- Clique: A complete subgraph

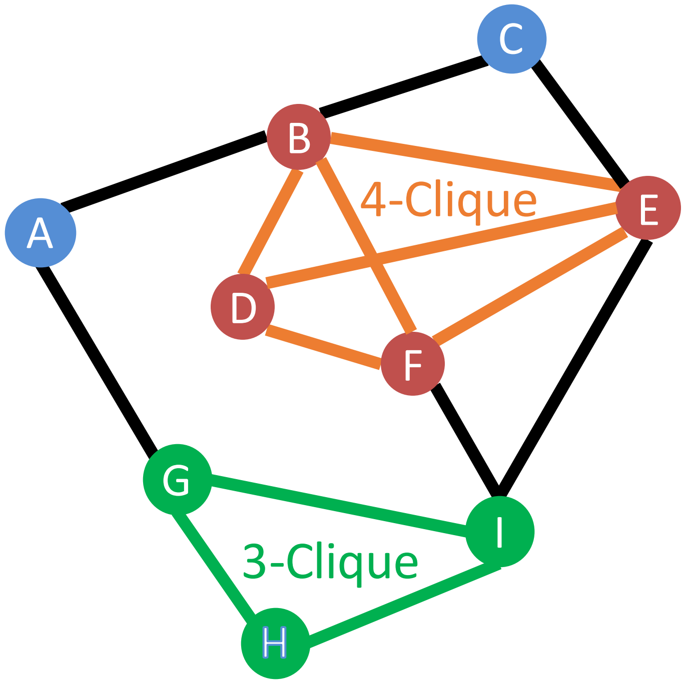


k-Clique is NP-Complete

1. Show that it belongs to NP
   – Give a polynomial time verifier
2. Show it is NP-Hard
   – Give a reduction from a known NP-Hard problem
   – We will show $3SAT \leq_p kClique$

k-Clique is in NP

- **Show:** For any graph G:
  - There is a short certificate ("solution") that G has a k-clique
  - The certificate can be checked efficiently (in polynomial time)



Graph G

Suppose $k = 4$

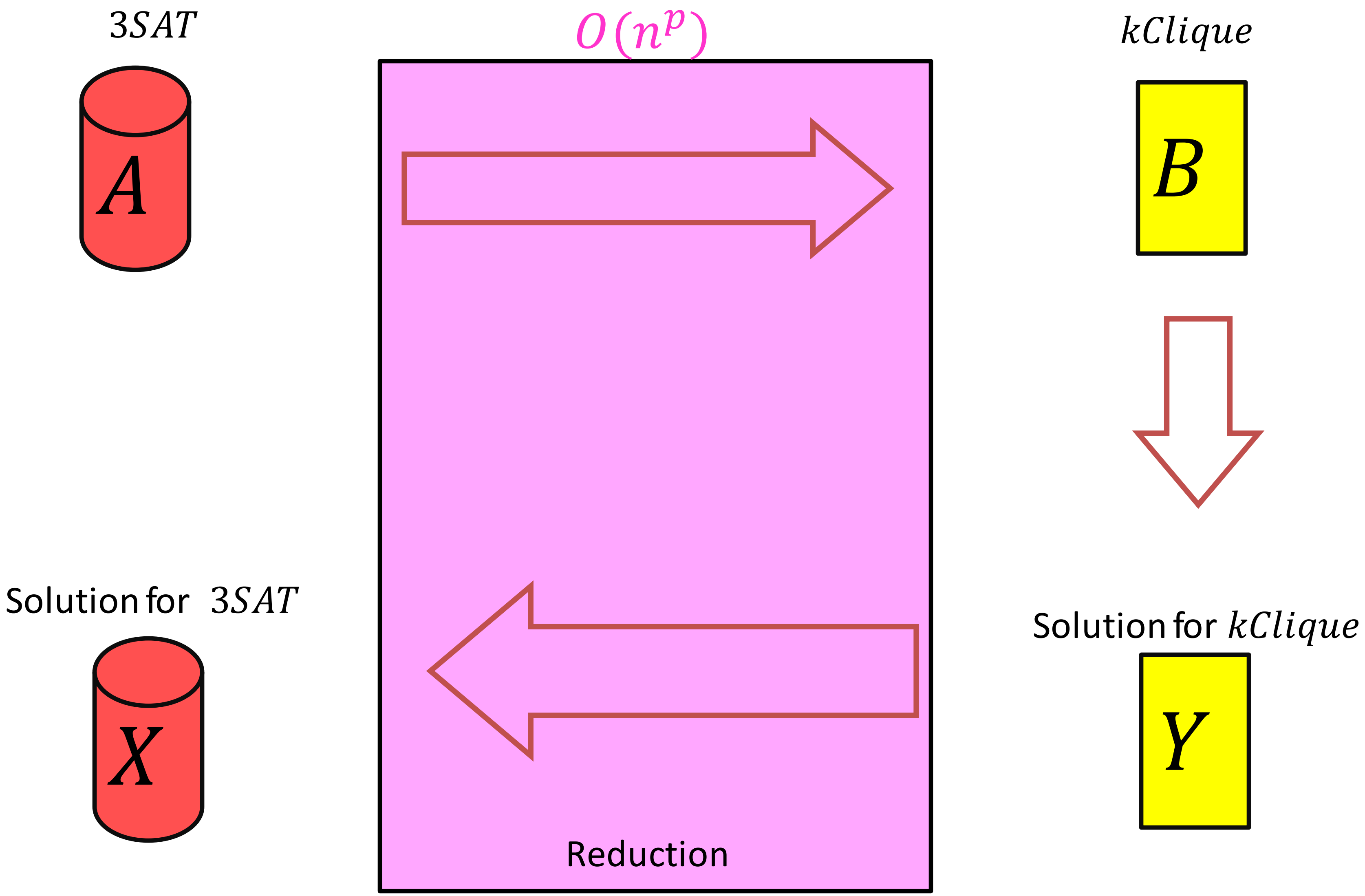
Certificate for *G*: $S = \{B, D, E, F\}$
(nodes in the k-clique)

Checking the certificate:

- Check that $|S| = k$ $\quad O(k) = O(|V|)$
- Check that every pair of nodes in S share an edge $\quad O(k^2) = O(|V|^2)$

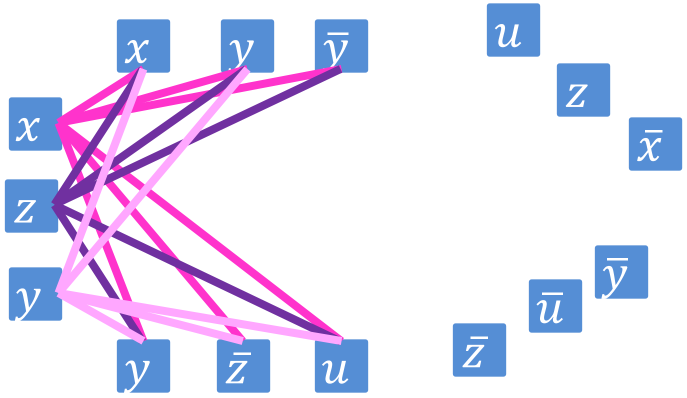
Total time: $O(|V|^2) = \text{poly}(|V| + |E|)$

$3SAT \leq_p kClique$



Instance of 3SAT to Instance of kClique

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



(also do this for the other clauses, omitted due to clutter)

For each clause, produce a node for each of its three variables

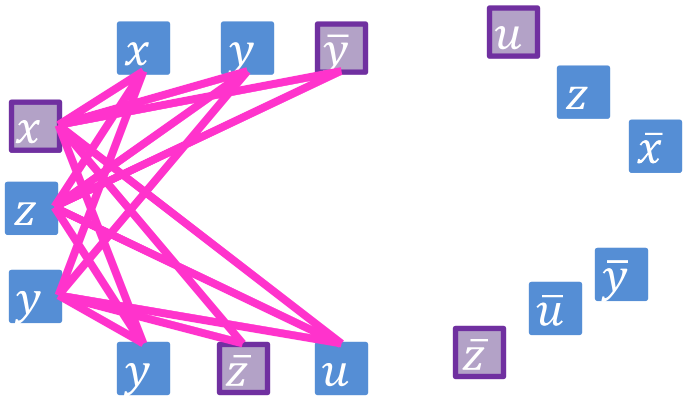
Connect each node to all non-contradictory nodes in the other clauses (i.e., anything that's not its negation)

Let k = number of clauses

There is a k-Clique in this graph **iff** there is a satisfying assignment

kClique $\Rightarrow$ Satisfying Assignment

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



$\boldsymbol{x = true}$
$\boldsymbol{y = false}$
$\boldsymbol{z = false}$
$\boldsymbol{u = true}$

There are k triplets in the graph, and no two nodes in the same triplet are adjacent

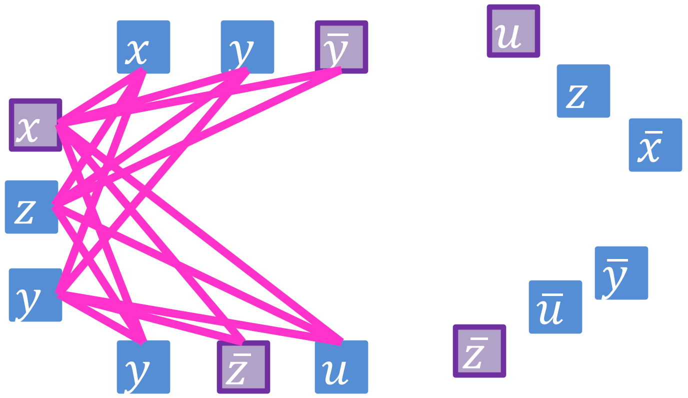
To have a k-Clique, must have one node from each triplet

Cannot select a node for both a variable and its negation

Therefore selection of nodes is a satisfying assignment

Satisfying Assignment $\Rightarrow k$Clique

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



$$\boldsymbol{x = true}$$
$$\boldsymbol{y = false}$$
$$\boldsymbol{z = false}$$
$$\boldsymbol{u = true}$$

Select one node for a true variable from each clause

There will be k nodes selected

We can't select both a node and its negation

All nodes will be non-contradictory, so they will be pairwise adjacent

$3SAT \leq_p kClique$