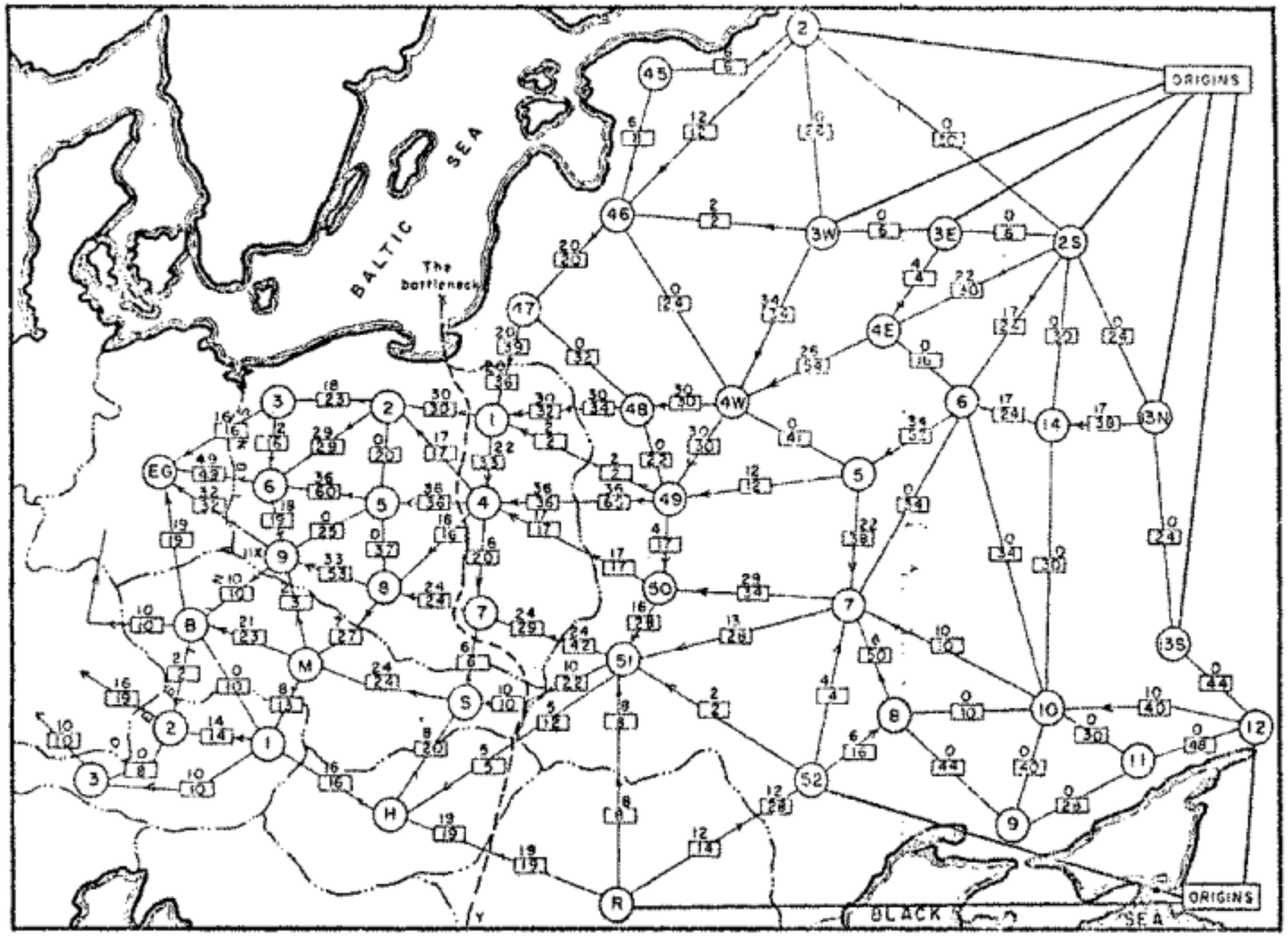
Network Flow



Railway map of Western USSR, 1955

Question: What is the maximum throughput of the railroad network?

Today's Keywords

- Max Flow, Min Cut
- Reductions
- Bipartite Matching
- Vertex Cover
- Independent Set

- CLRS Chapter 34

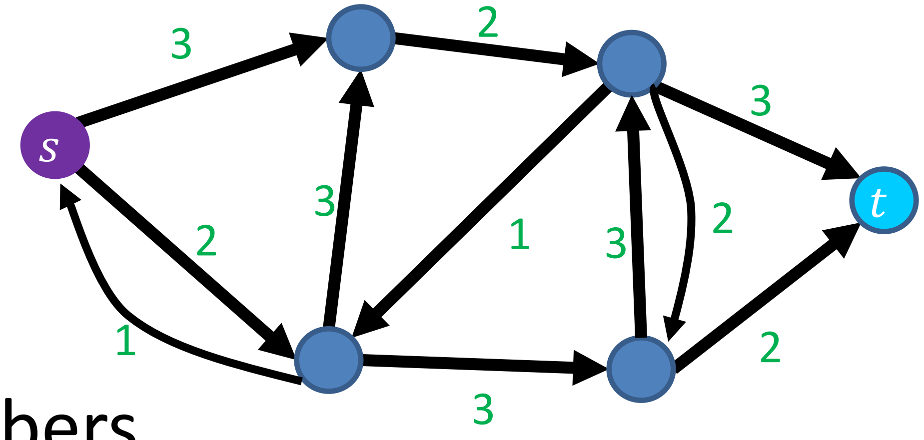
Flow Network

Graph $G = (V, E)$

Source node $s \in V$

Sink node $t \in V$

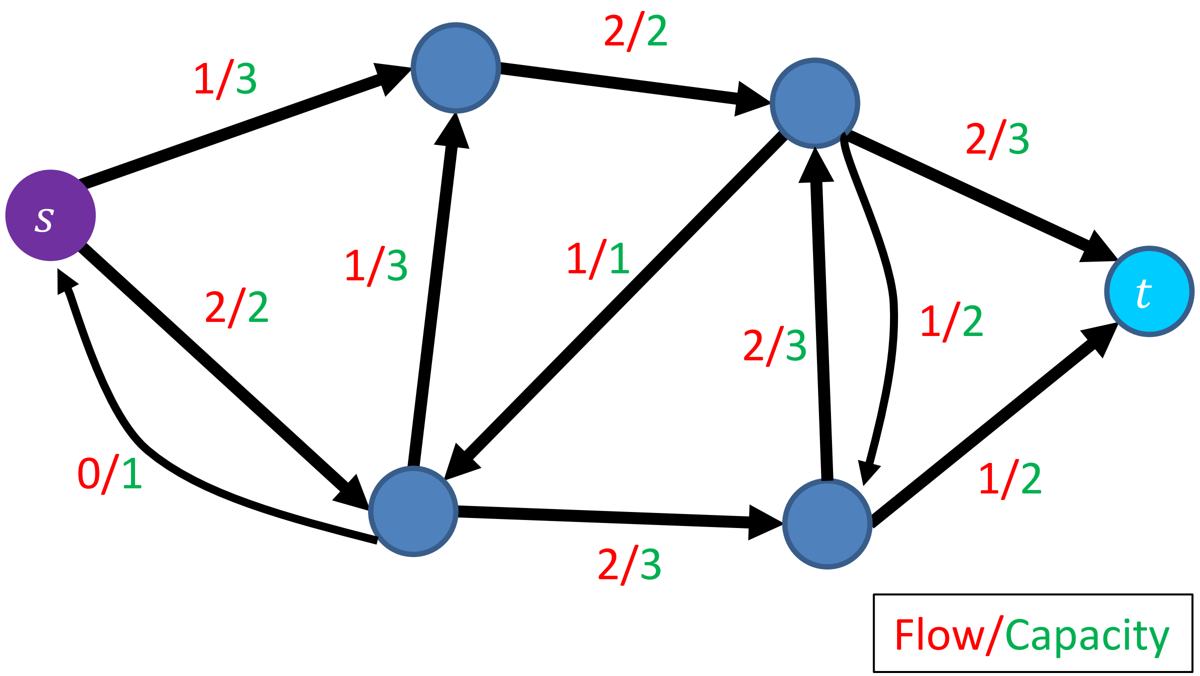
Edge Capacities $c(e) \in$ Positive Real numbers



Max flow intuition: If s is a faucet, t is a drain, and s connects to t through a network of pipes with given capacities, what is the maximum amount of water which can flow from the faucet to the drain?

Flow

- Assignment of values to edges
  - $f(e) = n$
  - Amount of water going through that pipe
- Capacity constraint
  - $f(e) \leq c(e)$
  - Flow cannot exceed capacity
- Flow constraint
  - $\forall v \in V - \{s, t\}, inflow(v) = outflow(v)$
  - $inflow(v) = \sum_{x \in V} f(x, v)$
  - $outflow(v) = \sum_{x \in V} f(v, x)$
  - Water going in must match water coming out
- Flow of G: $|f| = outflow(s) - inflow(s)$
  - Net outflow of s



3 in example above

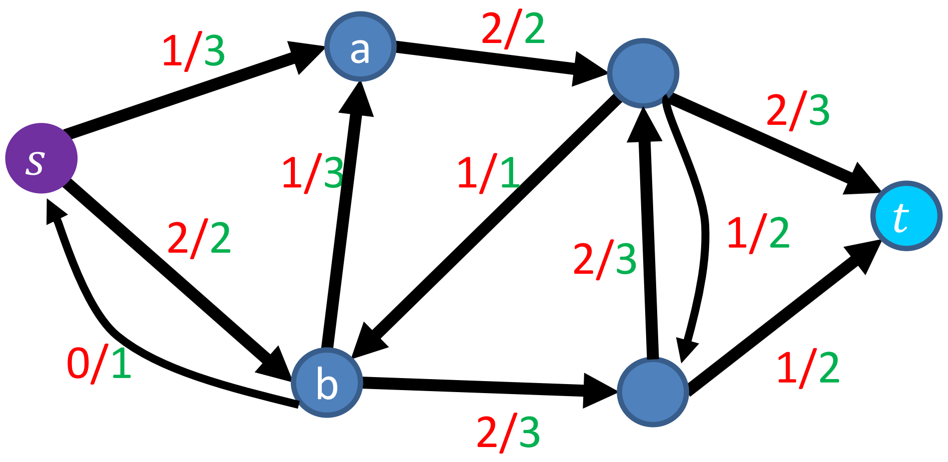
Max Flow

- Of all valid flows through the graph, find the one which maximizes:
  - $|f| = outflow(s) - inflow(s)$

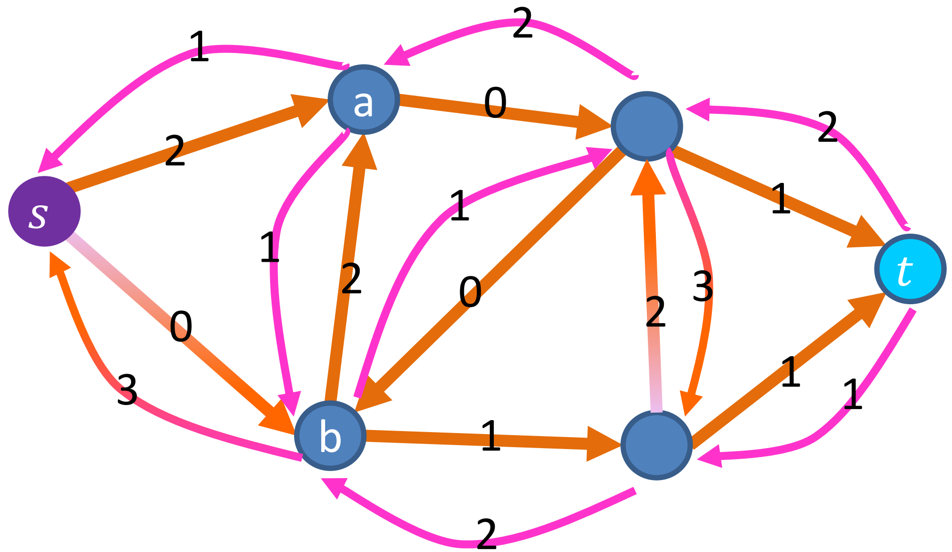
Residual Graph G_f

- Keep track of net available flow along each edge
- Forward edges: weight is equal to available flow along that edge in the flow graph
  Flow I *could* add
  - $w(e) = c(e) - f(e)$
- Back edges: weight is equal to flow along that edge in the flow graph
  Flow I *could* remove
  - $w(e) = f(e)$

Flow Graph G

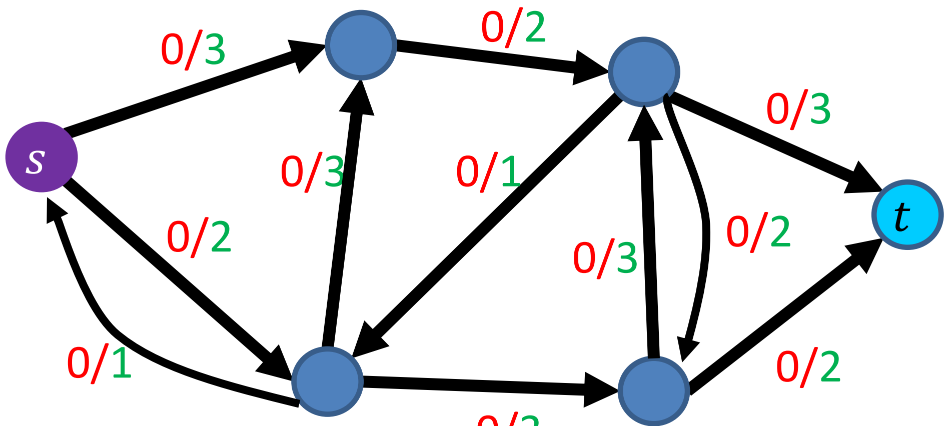


Residual Graph G_f



Ford Fulkerson: example

Flow Graph G



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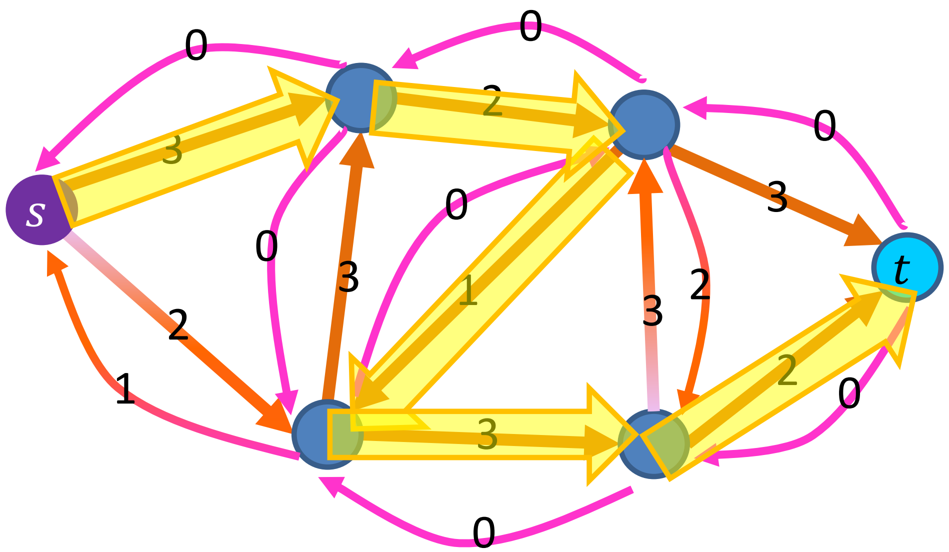
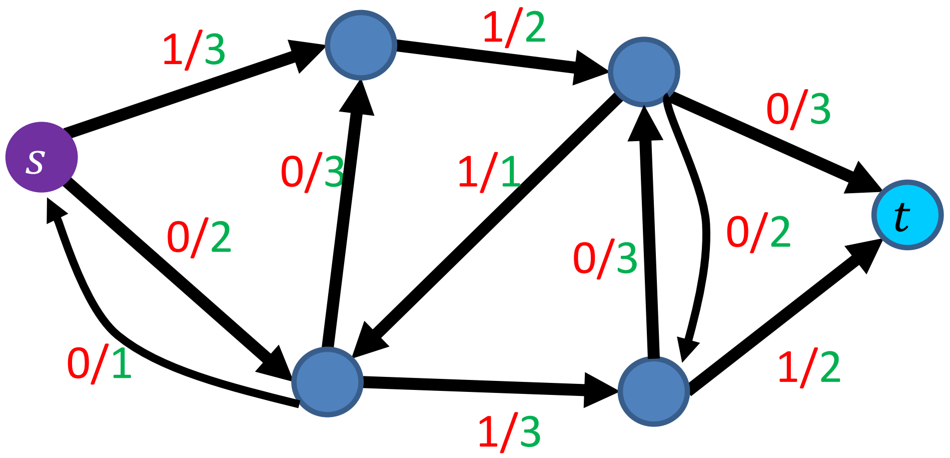


Add flow of 1 to this path

Ford Fulkerson: example

Flow Graph G



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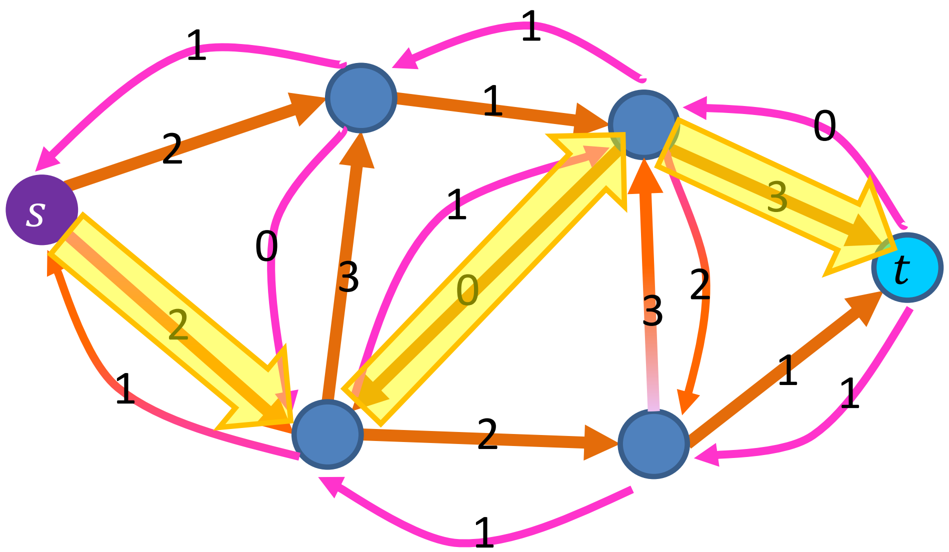

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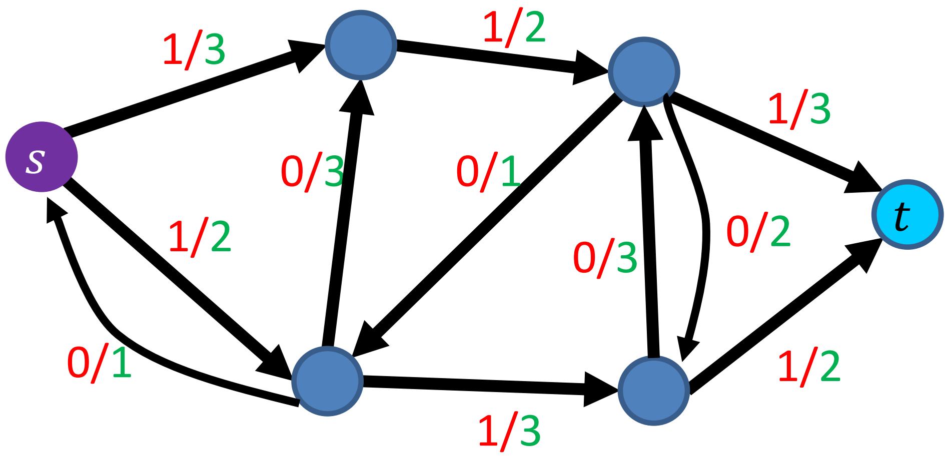
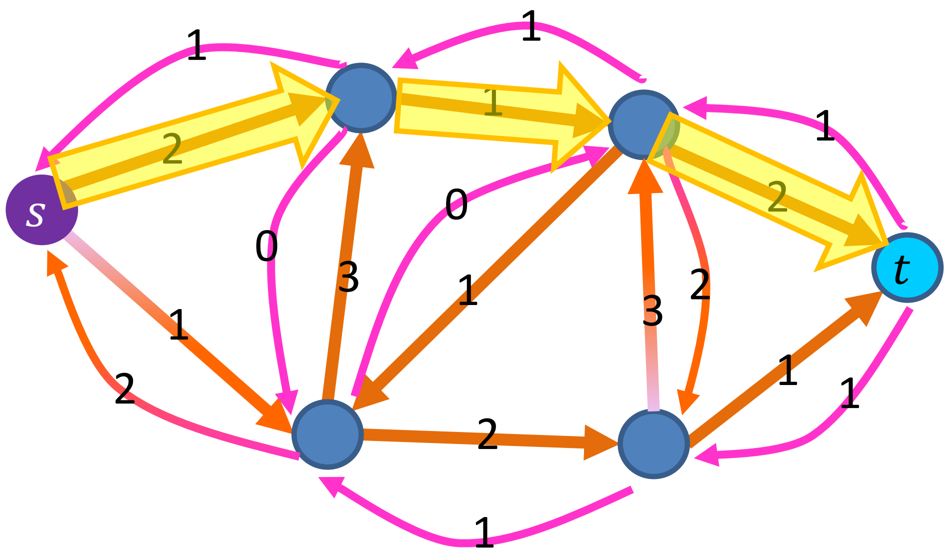


Residual Graph G_f



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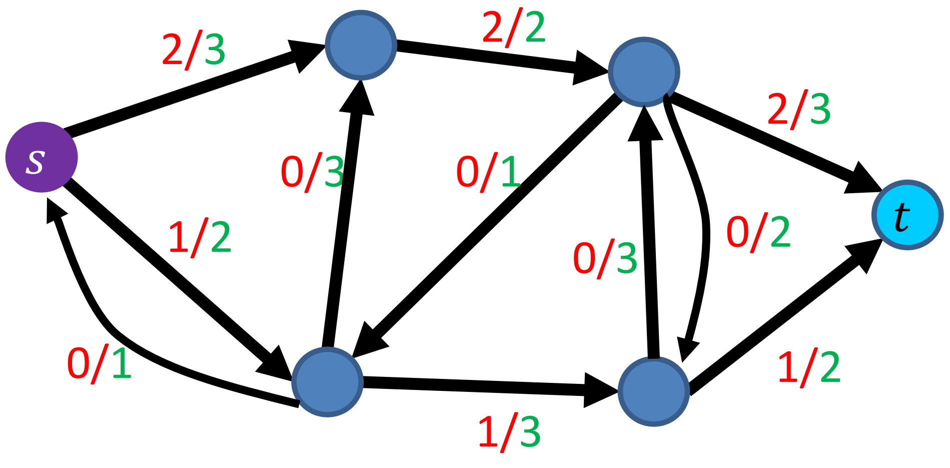

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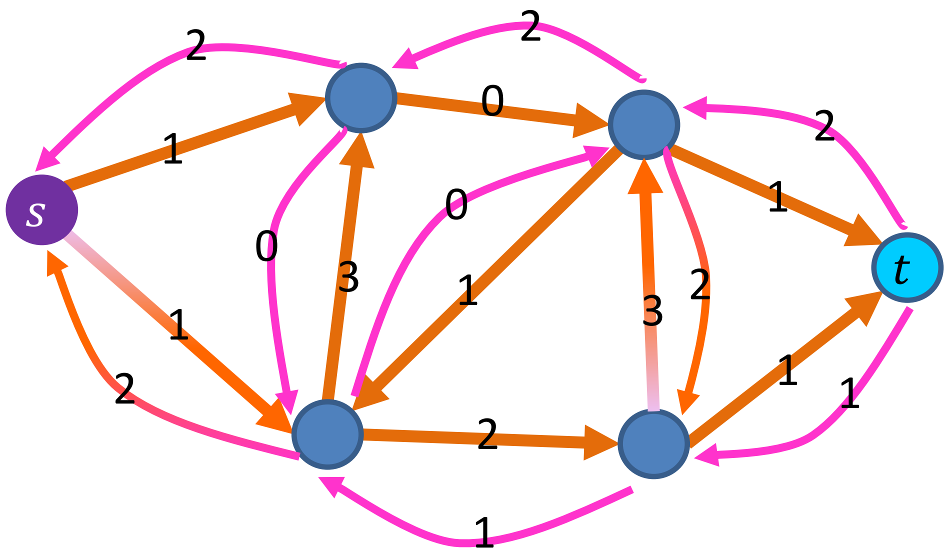

Ford-Fulkerson Running Time

Define an augmenting path to be an $s \to t$ path in the residual graph G_f (using edges of non-zero weight)

Ford-Fulkerson max-flow algorithm:

- Initialize $f(e) = 0$ for all $e \in E$
- Construct the residual network G_f
- While there is an augmenting path p in G_f:
  - Let $c = \min_{e \in E} c_f(e)$ ($c_f(e)$ is the weight of edge e in the residual network G_f)
  - Add c units of flow to G based on the augmenting path p
  - Update the residual network G_f for the updated flow

Initialization: $O(|E|)$

Ford-Fulkerson Running Time

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Initialization: $O(|E|)$

Construct residual network: $O(|E|)$

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We only care about nodes reachable from the source s (so the number of nodes that are "relevant" is at most $|E|$)

Initialization: $O(|E|)$

Construct residual network: $O(|E|)$

Finding augmenting path in residual network: $O(|E|)$ using BFS/DFS

Ford-Fulkerson Running Time

Define an augmenting path to be an $s \to t$ path in the residual graph G_f (using edges of non-zero weight)

How many iterations are needed?

- For integer-valued capacities, min-weight of each augmenting path is 1, so number of iterations is bounded by $|f^*|$, where $|f^*|$ is max-flow in G
- For rational-valued capacities, can scale to make capacities integer
- For irrational-valued capacities, algorithm may never terminate!

Initialization: $O(|E|)$
Construct residual network: $O(|E|)$
Finding augmenting path in residual network: $O(|E|)$ using BFS/DFS

Ford-Fulkerson Running Time

Define an augmenting path to be an $s \to t$ path in the residual graph G_f (using edges of non-zero weight)

Ford-Fulkerson max-flow algorith

- Initialize $f(e) = 0$ for all e
- Construct the residual net
- While there is an augmen
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  - Add c units of flow to
  - Update the residual n

Initialization: $O(|E|)$

Construct residual network: $O(|E|)$

Finding augmenting path in residual network: $O(|E|)$ using BFS/DFS

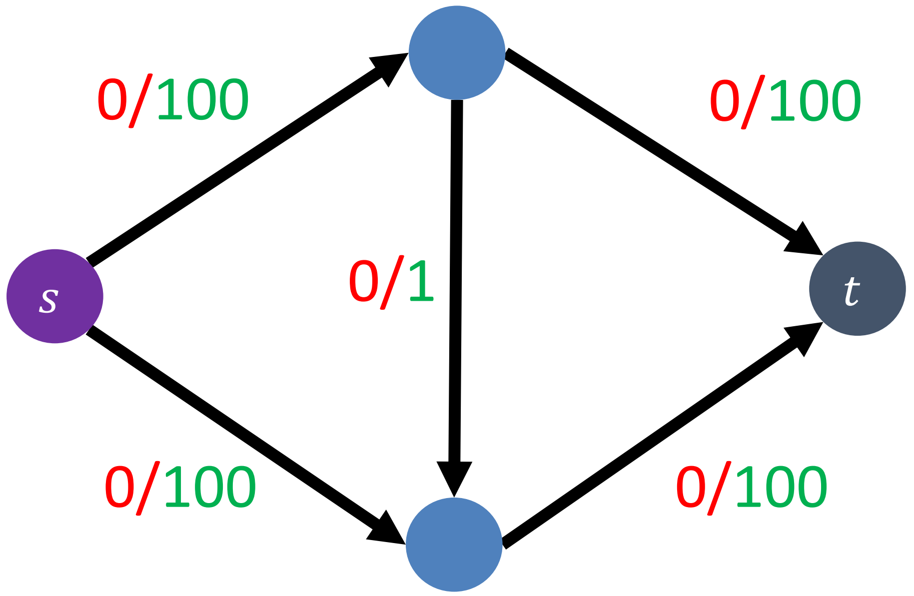
For graphs with integer capacities, running time of Ford-Fulkerson is

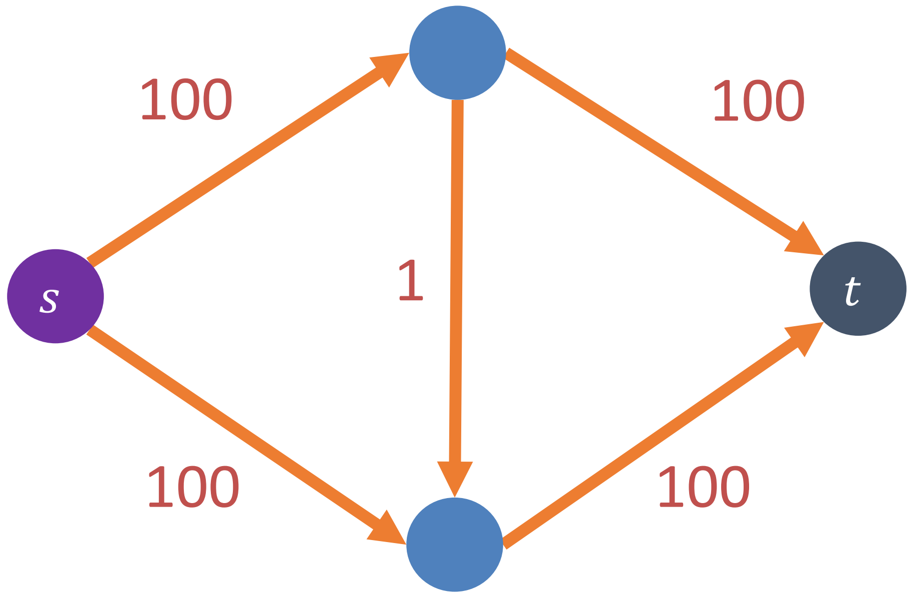
$$O(|f^*| \cdot |E|)$$

Highly undesirable if $|f^*| \gg |E|$ (e.g., graph is small, but capacities are $\approx 2^{32}$)

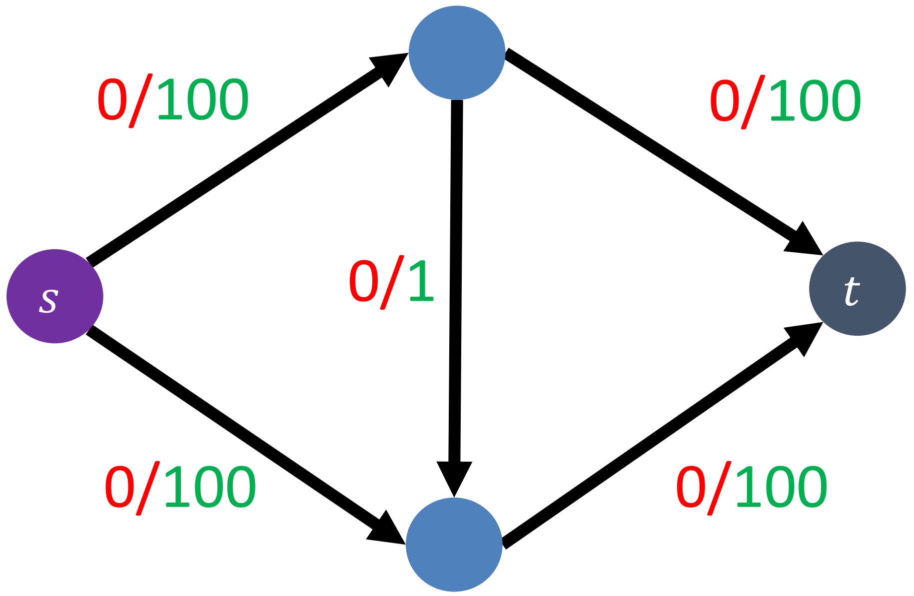
As described, algorithm is <u>not</u> polynomial-time!

Worst-Case Ford-Fulkerson

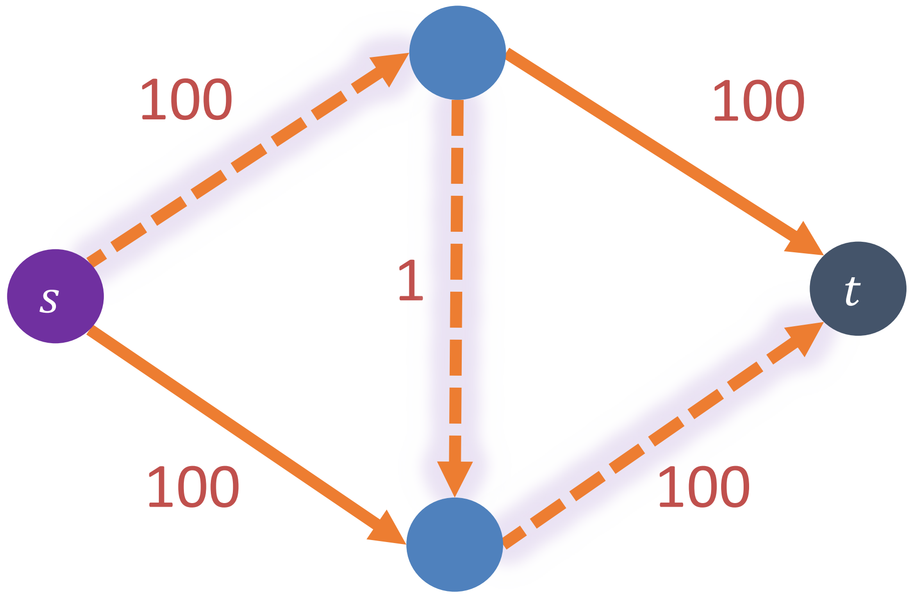




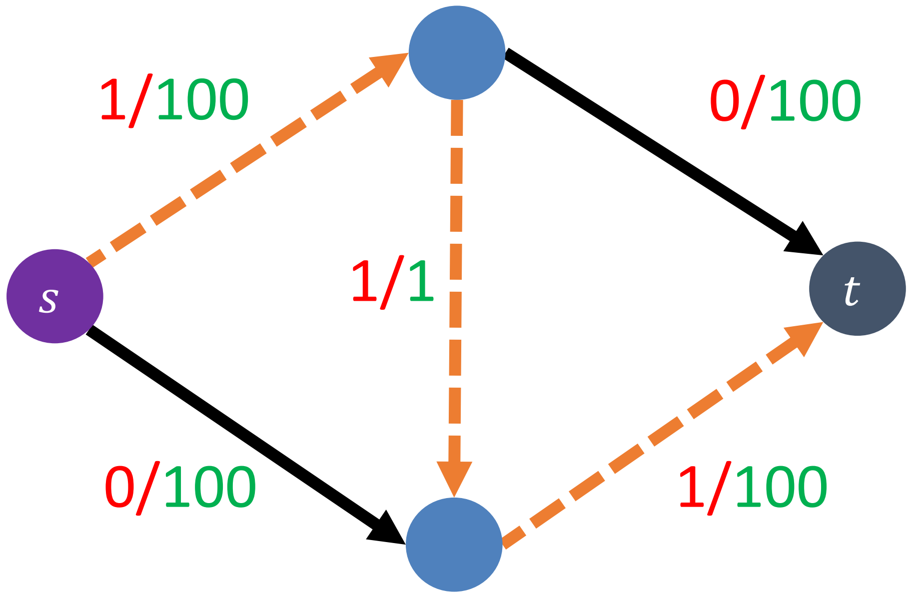
Worst-Case Ford-Fulkerson



Increase flow by 1 unit



Worst-Case Ford-Fulkerson



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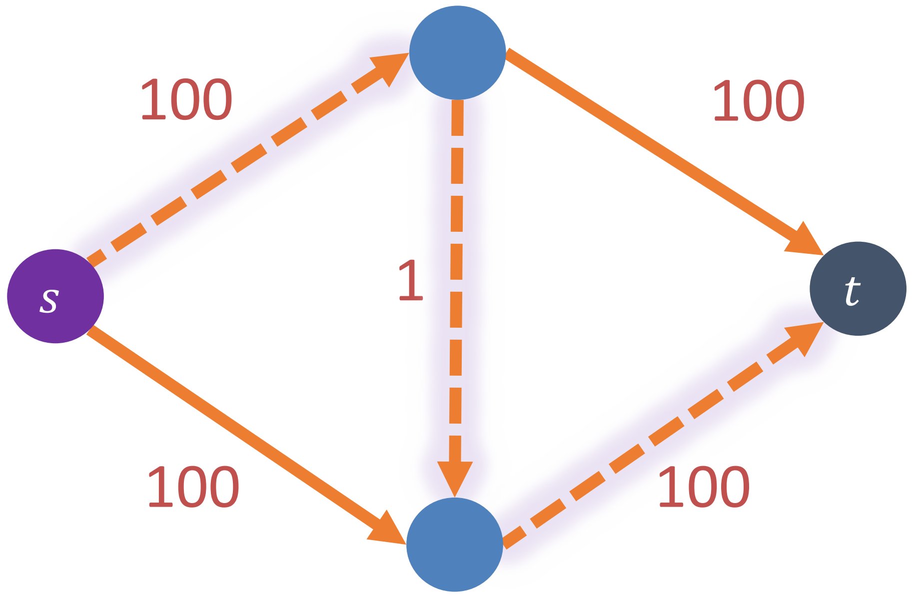

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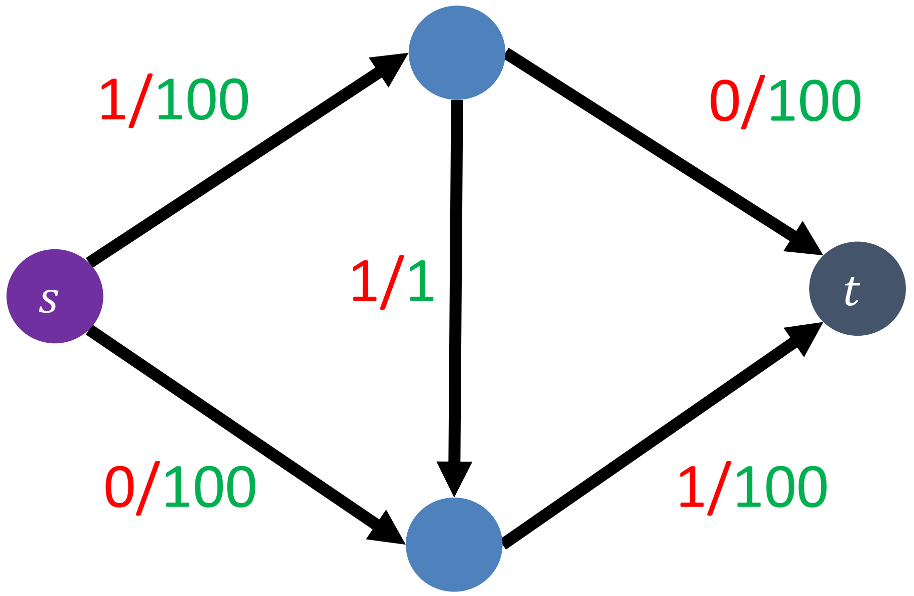
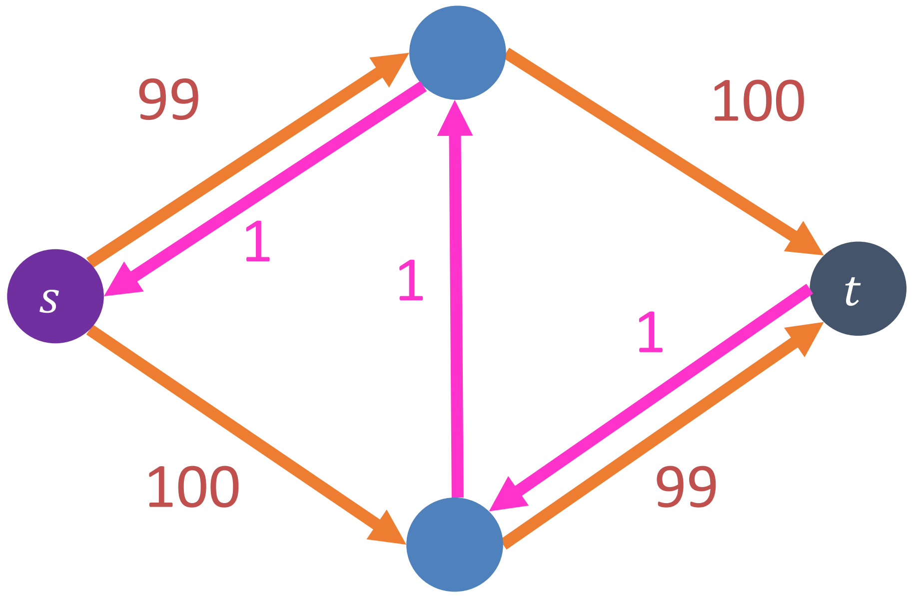
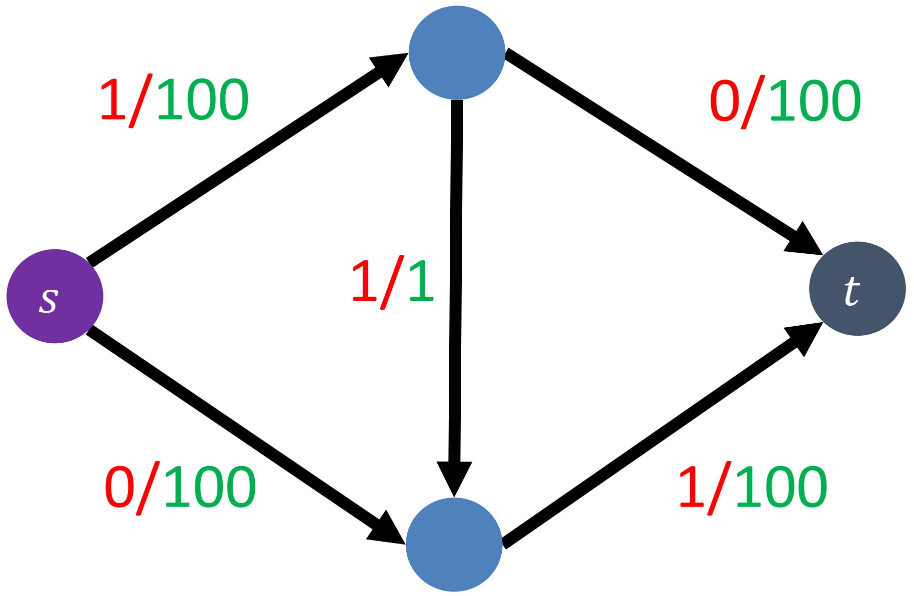




Worst-Case Ford-Fulkerson



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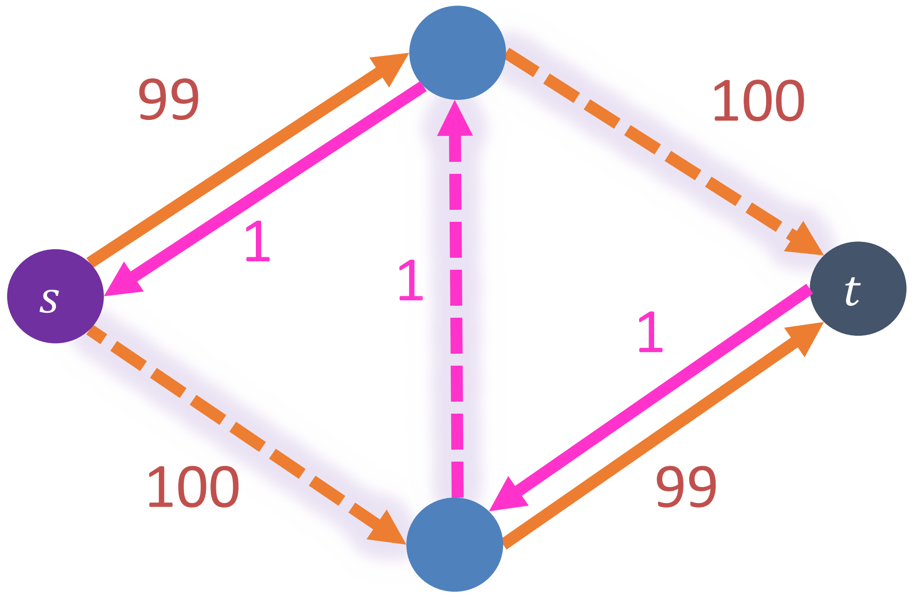

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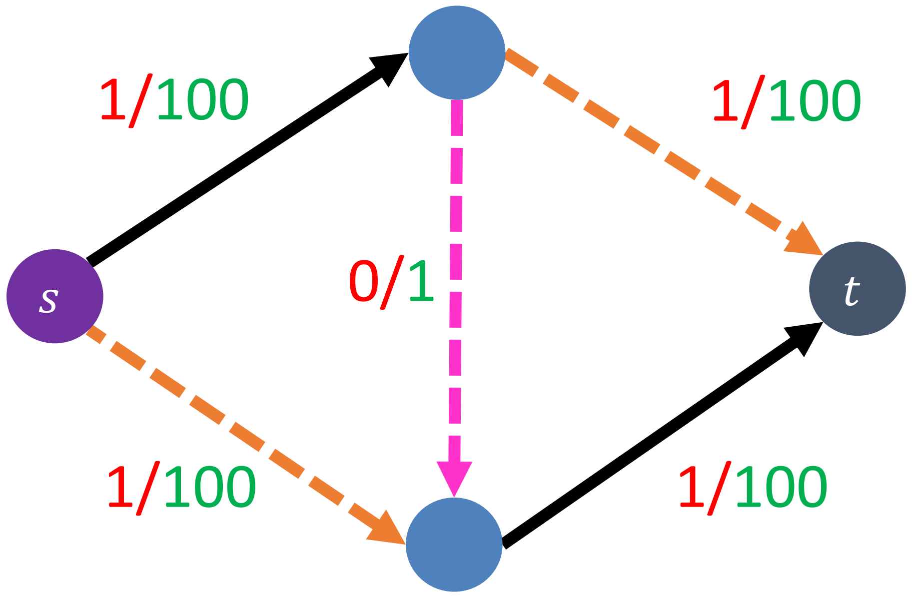
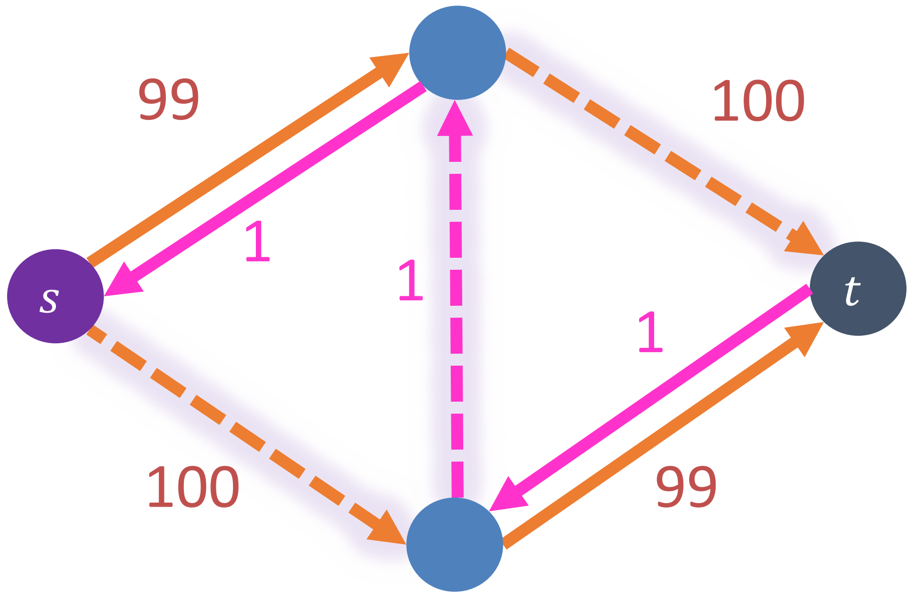


Increase flow by 1 unit



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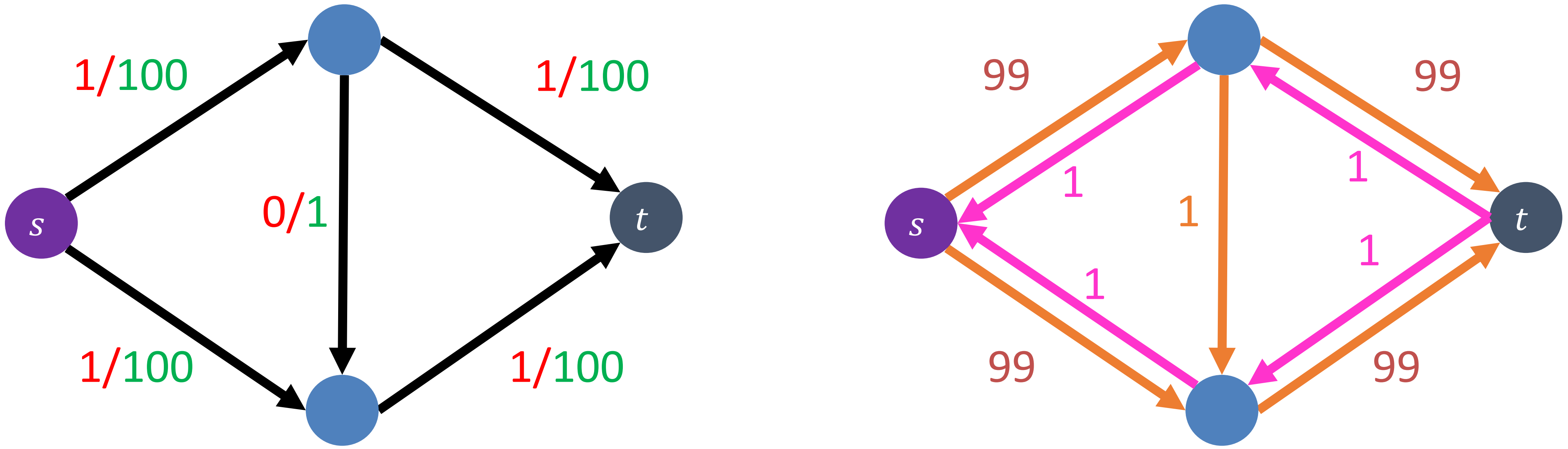

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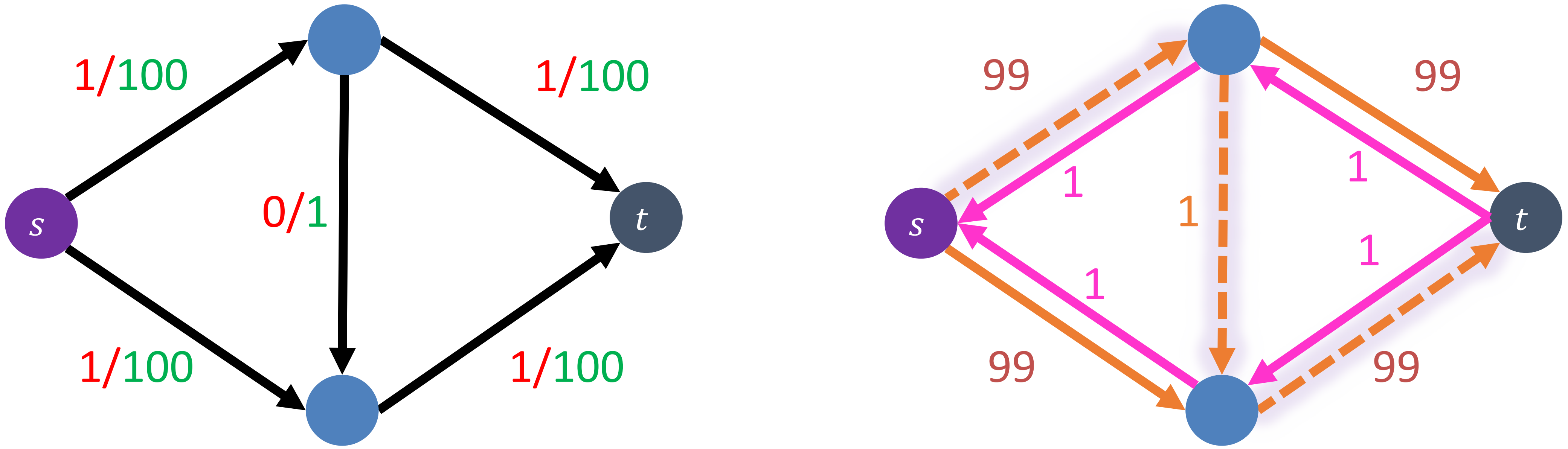


Observation: each iteration increases flow by 1 unit

Total number of iterations: $|f^*| = 200$

Can We Avoid this?

- **Edmonds-Karp Algorithm:** choose augmenting path with fewest hops
- **Running time:** $\Theta(\min(|E||f^*|, |V||E|^2))$

How to find this?
Use breadth-first search (BFS)!

Edmonds-Karp = Ford-Fulkerson using BFS to find augmenting path

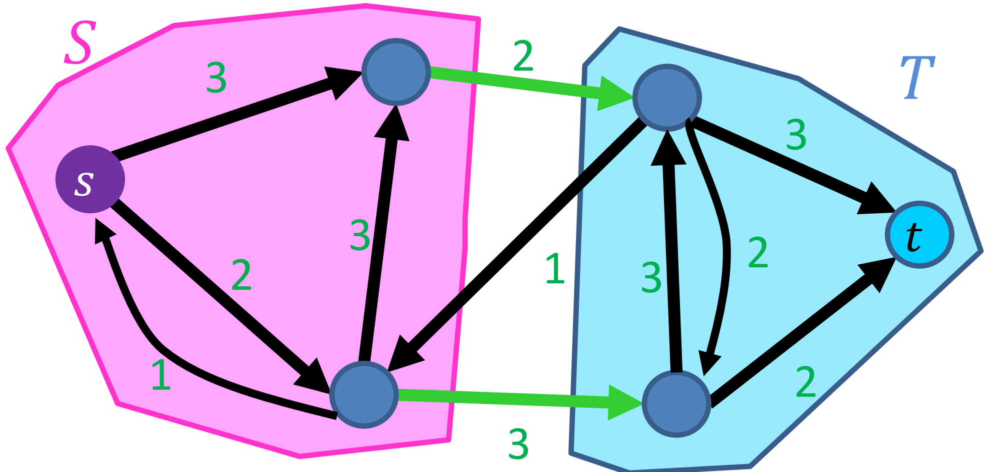
Ford-Fulkerson max-flow algorithm:

- Initialize $f(e) = 0$ for all $e \in E$
- Construct the residual network G_f
- While there is an augmenting path in G_f, let p be the path with fewest hops:
  - Let $c = \min_{e \in E} c_f(e)$ ($c_f(e)$ is the weight of edge e in the residual network G_f)
  - Add c units of flow to G based on the augmenting path p
  - Update the residual network G_f for the updated flow

Proof: See CLRS (Chapter 26.2)

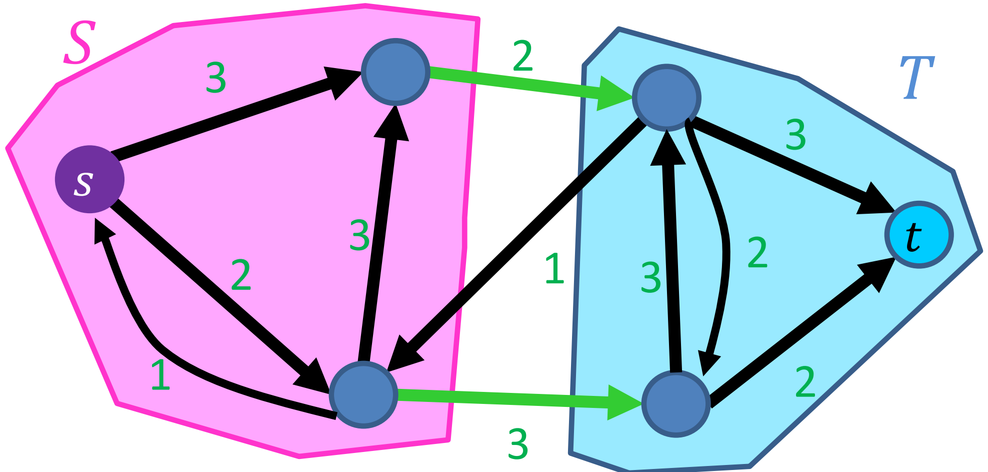
Showing Correctness of Ford-Fulkerson

- Consider cuts which separate s and t
  - Let $s \in S$, $t \in T$, s.t. $V = S \cup T$
- Cost of cut $(S, T) = ||S, T||$
  - Sum **capacities** of edges which go from S to T
  - This example: 5



Maxflow≤MinCut

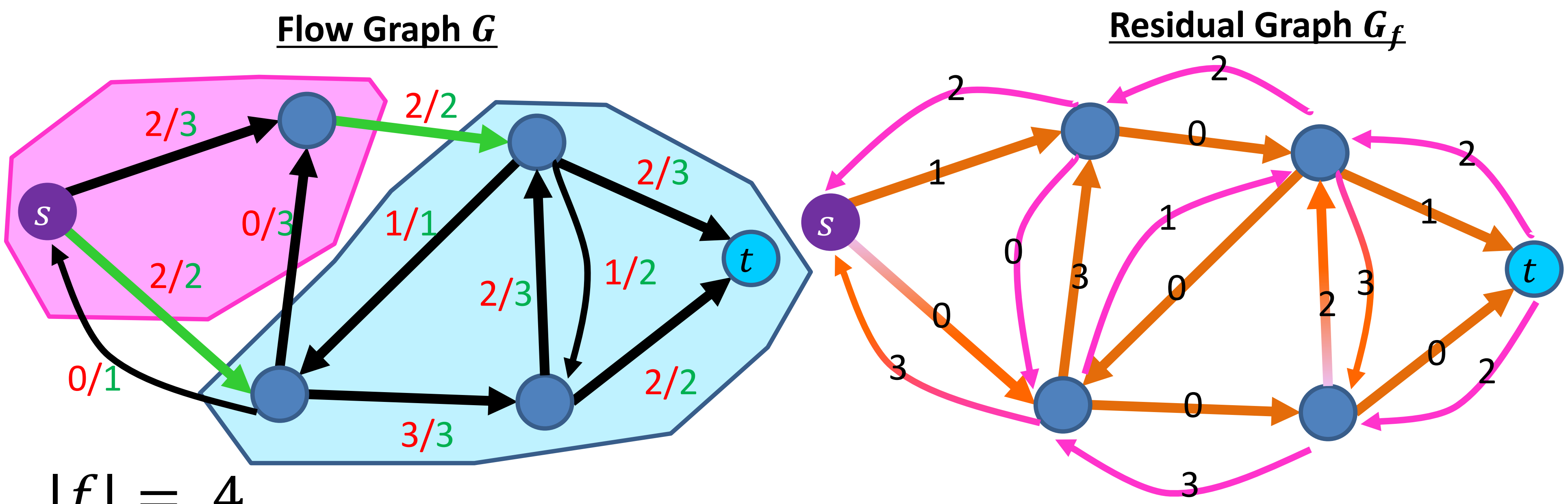
- Max flow upper bounded by any cut separating s and t
- Why? "Conservation of flow"
  - All flow exiting s must eventually get to t
  - To get from s to t, all "tanks" must cross the cut
- Conclusion: If we find the minimum-cost cut, we've found the maximum flow
  - $\max_f |f| \le \min_{S,T} ||S,T||$



Maxflow/Mincut Theorem

- To show Ford-Fulkerson is correct:
  - Show that when there are no more augmenting paths, there is a cut with cost equal to the flow
- Conclusion: the maximum flow through a network matches the minimum-cost cut
  - $\max_{f}|f| = \min_{S,T}||S,T||$
- Duality
  - When we've maximized max flow, we've minimized min cut (and vice-versa), so we can check when we've found one by finding the other

Example: Maxflow/Mincut



$|f| = 4$

$||S, T|| = 4$

No Augmenting Paths

Idea: When there are no more augmenting paths, there exists a cut in the graph with cost matching the flow

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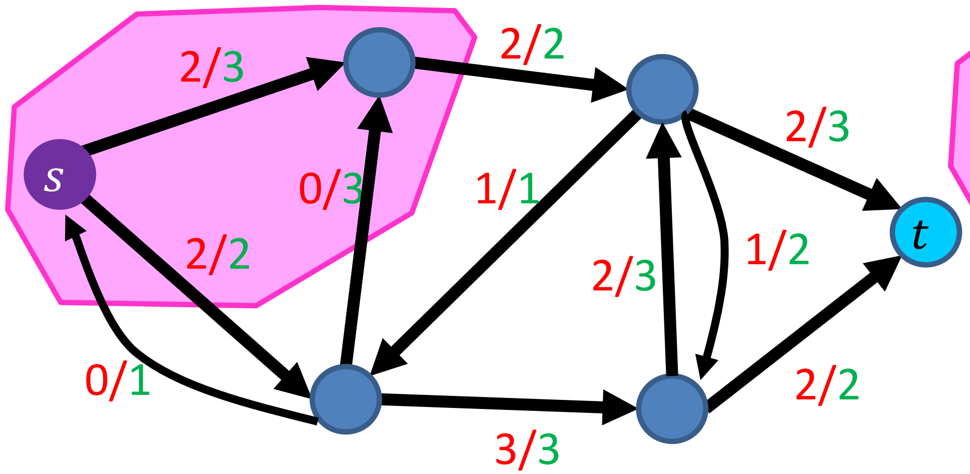
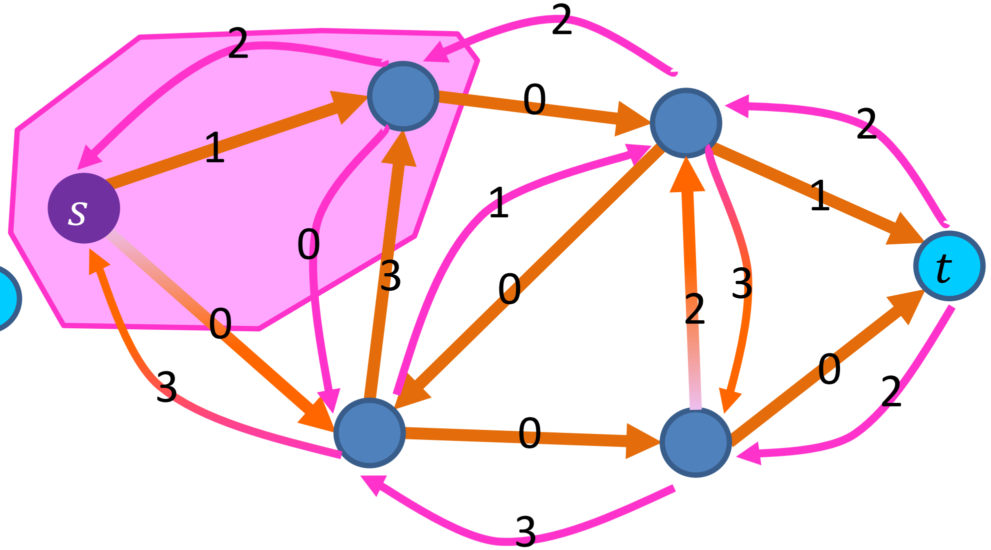
Proof: Maxflow/Mincut Theorem

- If $|f|$ is a max flow, then G_f has no augmenting path
  - Otherwise, use that augmenting path to "push" more flow
- Define S = nodes reachable from source node s by positive-weight edges in the residual graph
  - $T = V - S$
  - S separates s, t (otherwise there's an augmenting path)

Flow Graph G



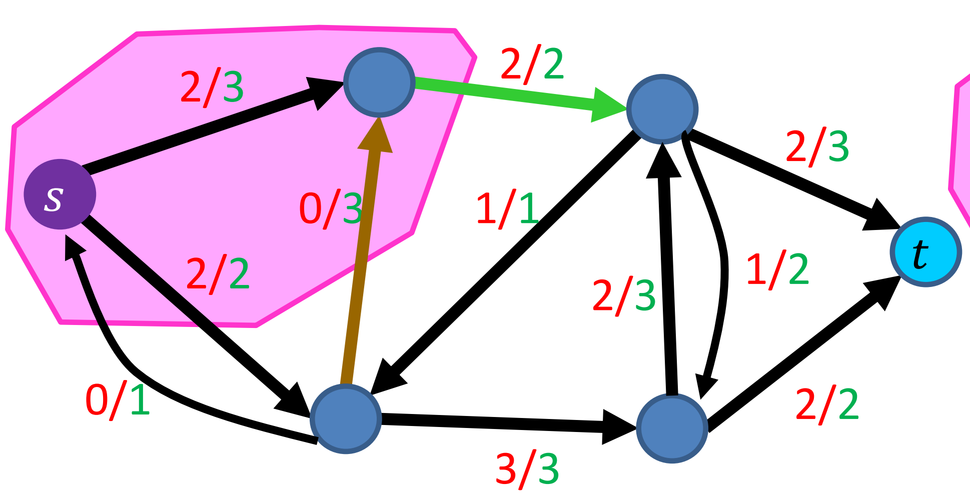
Residual Graph G_f



Proof: Maxflow/Mincut Theorem

- To show: $||S, T|| = |f|$
  - Weight of the cut matches the flow across the cut
- Consider edge (u, v) with $u \in S, v \in T$
  - $f(u, v) = c(u, v)$, because otherwise $w(u, v) > 0$ in G_f, which would mean $v \in S$
- Consider edge (y, x) with $y \in T, x \in S$
  - $f(y, x) = 0$, because otherwise the back edge $w(y, x) > 0$ in G_f, which would mean y $\in S$

Flow Graph G



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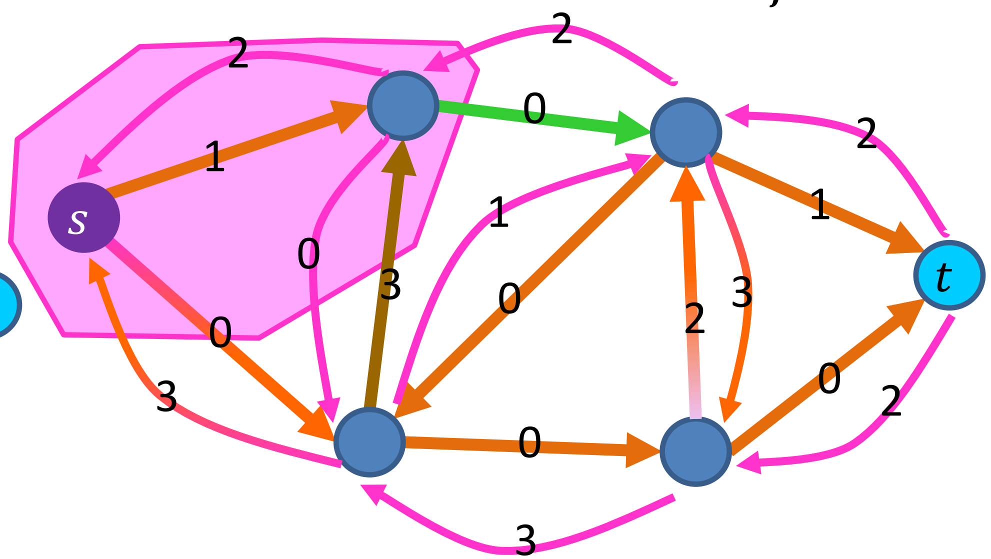


Proof Summary

1. The flow $|f|$ of G is upper-bounded by the sum of capacities of edges crossing any cut separating source s and sink t

2. When Ford-Fulkerson terminates, there are no more augmenting paths in G_f

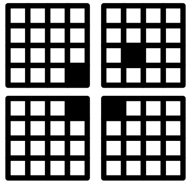
3. When there are no more augmenting paths in G_f then we can define a cut $S =$ nodes reachable from source node s by positive-weight edges in the residual graph

4. The sum of edge capacities crossing this cut must match the flow of the graph

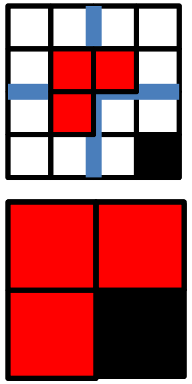
5. Therefore this flow is maximal

Divide and Conquer*

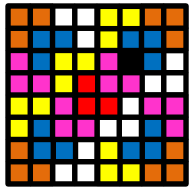
- **Divide**:



  – Break the problem into multiple subproblems, each smaller instances of the original
- **Conquer**:



  – If the suproblems are "large":
    - Solve each subproblem recursively
  – If the subproblems are "small":
    - Solve them directly (base case)
- **Combine**:



  – Merge together solutions to subproblems

*CLRS Chapter 4

Dynamic Programming

- Requires Optimal Substructure
  - – Solution to larger problem contains the solutions to smaller ones
- Idea:
  1. Identify recursive structure of the problem
  2. Select a good order for solving subproblems
     - Usually smallest problem first

Greedy Algorithms

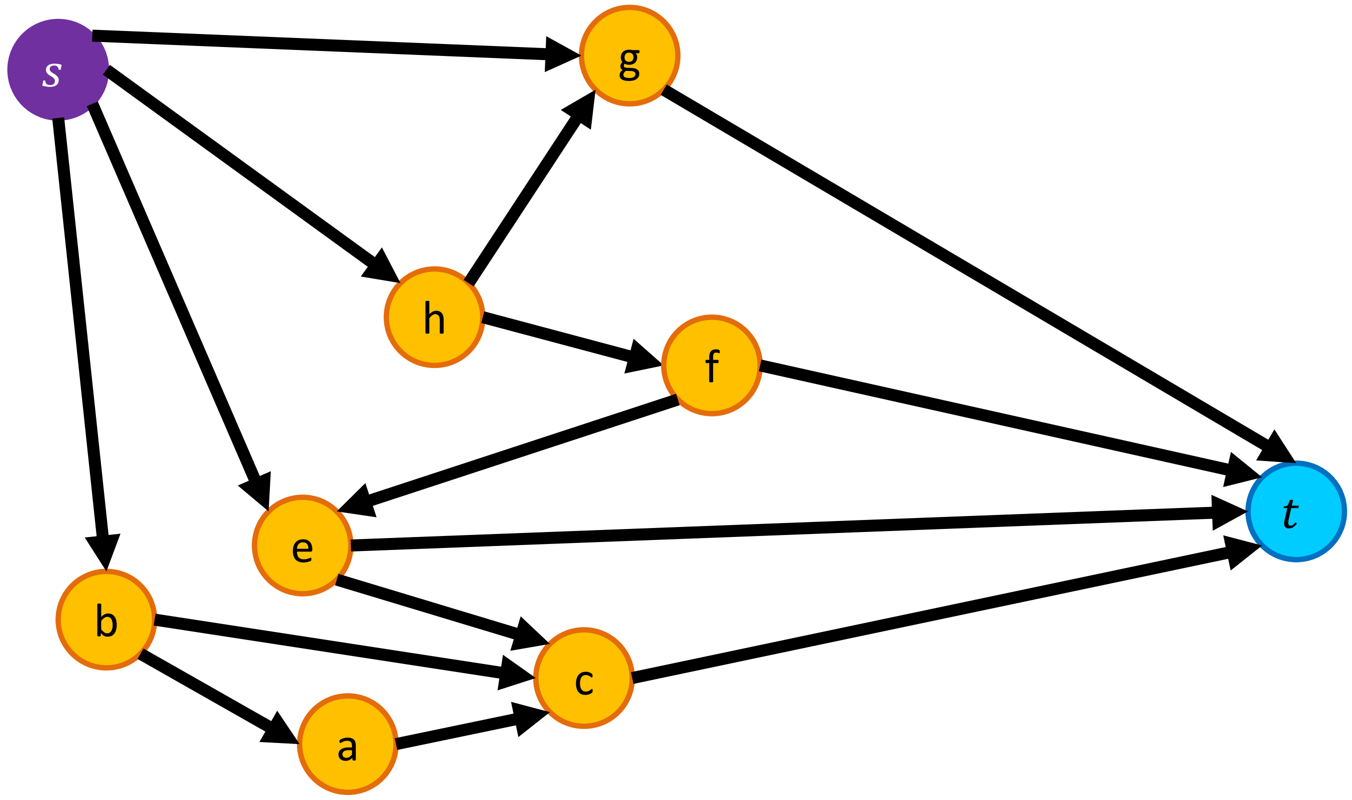
- Require Optimal Substructure
  - Solution to larger problem contains the solution to a smaller one
  - Only one subproblem to consider!
- Idea:
  1. Identify a greedy choice property
     - How to make a choice guaranteed to be included in some optimal solution
  2. Repeatedly apply the choice property until no subproblems remain

So far

- Divide and Conquer, Dynamic Programming, Greedy
  - Take an instance of Problem A, relate it to smaller instances of Problem A
- Next:
  - Take an instance of Problem A, relate it to an instance of Problem B

Edge-Disjoint Paths

Given a graph $G = (V, E)$, a start node s and a destination node t, give the maximum number of paths from s to t which share no edges



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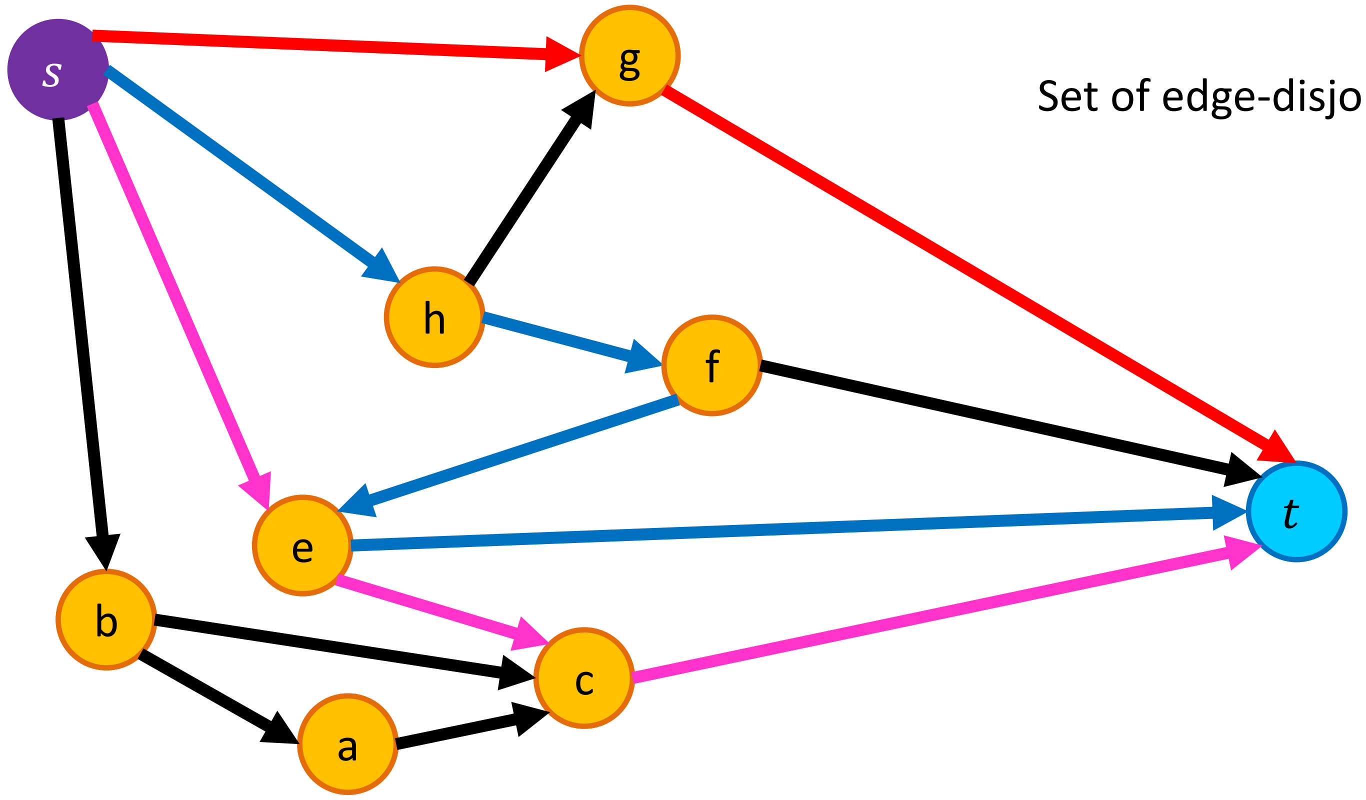
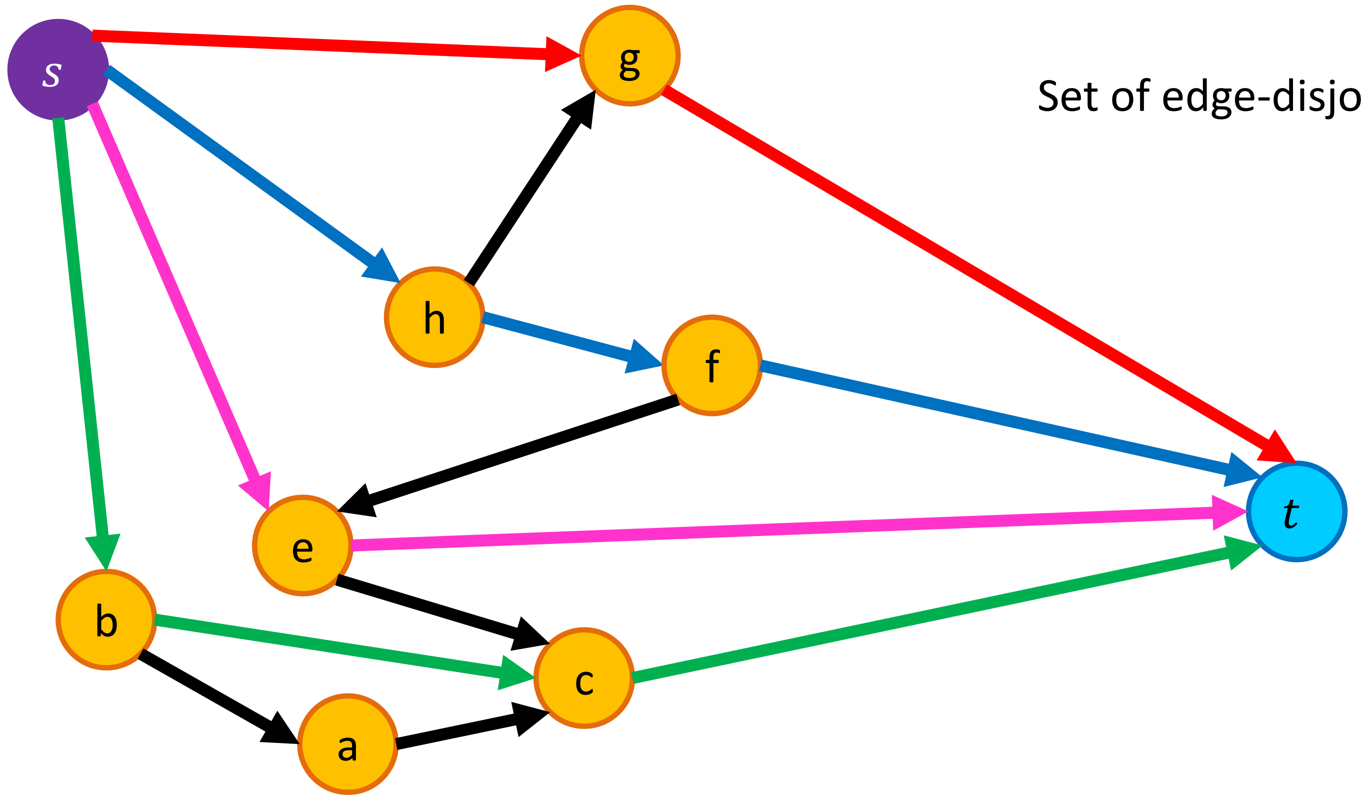


Set of edge-disjoint paths of size 3

Edge-Disjoint Paths

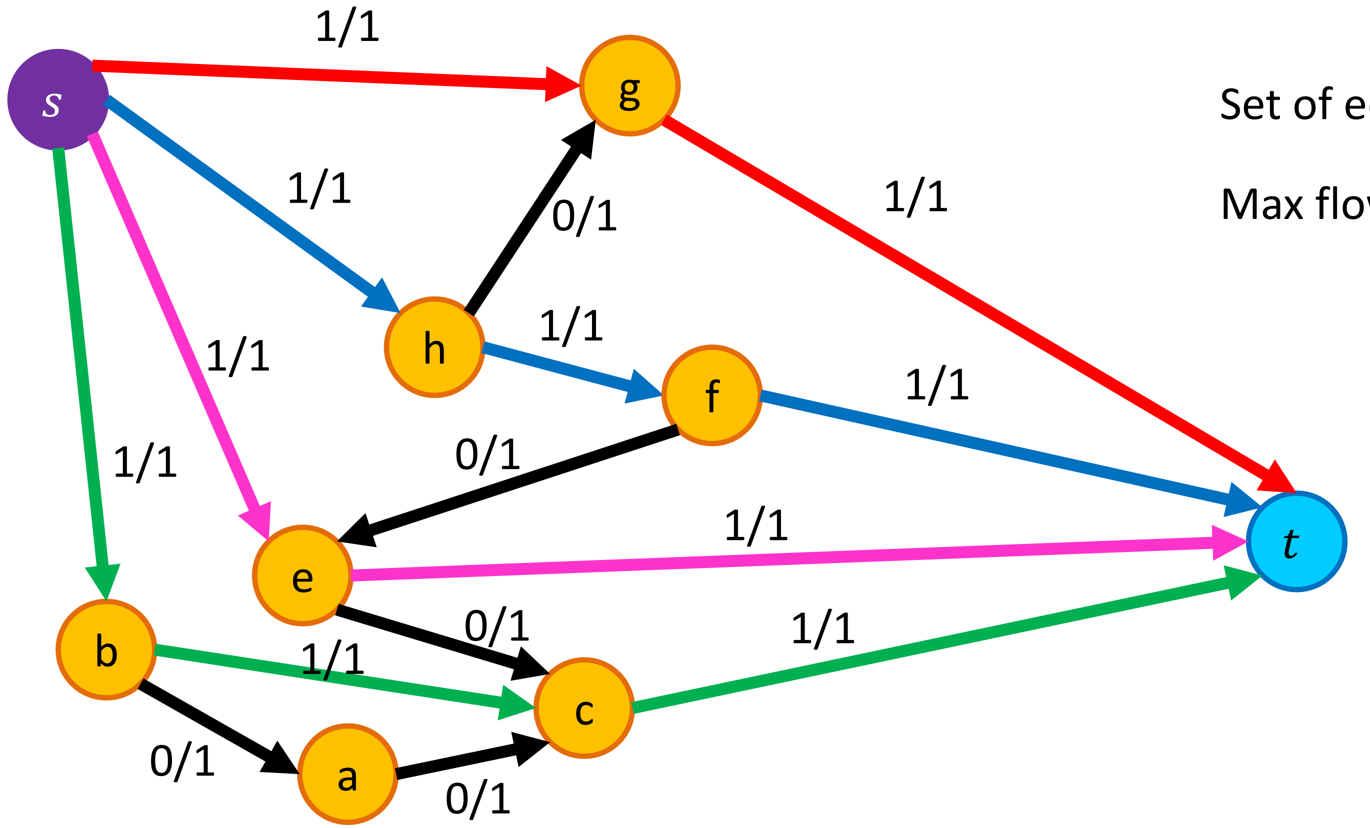
Given a graph $G = (V, E)$, a start node s and a destination node t, give the maximum number of paths from s to t which share no edges



Set of edge-disjoint paths of size 4

Edge-Disjoint Paths Algorithm

Make s and t the source and sink, give each edge capacity 1, find the max flow.

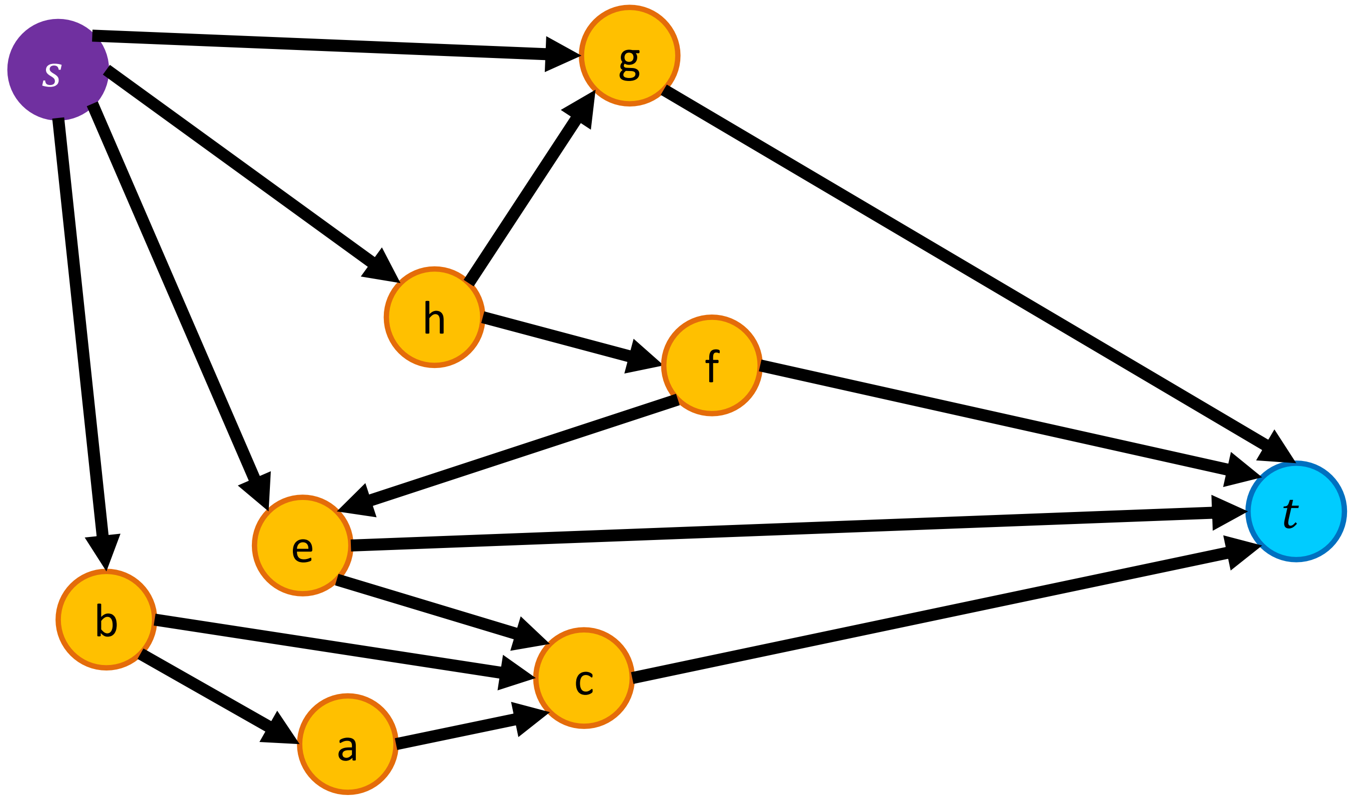


Set of edge-disjoint paths of size 4

Max flow = 4

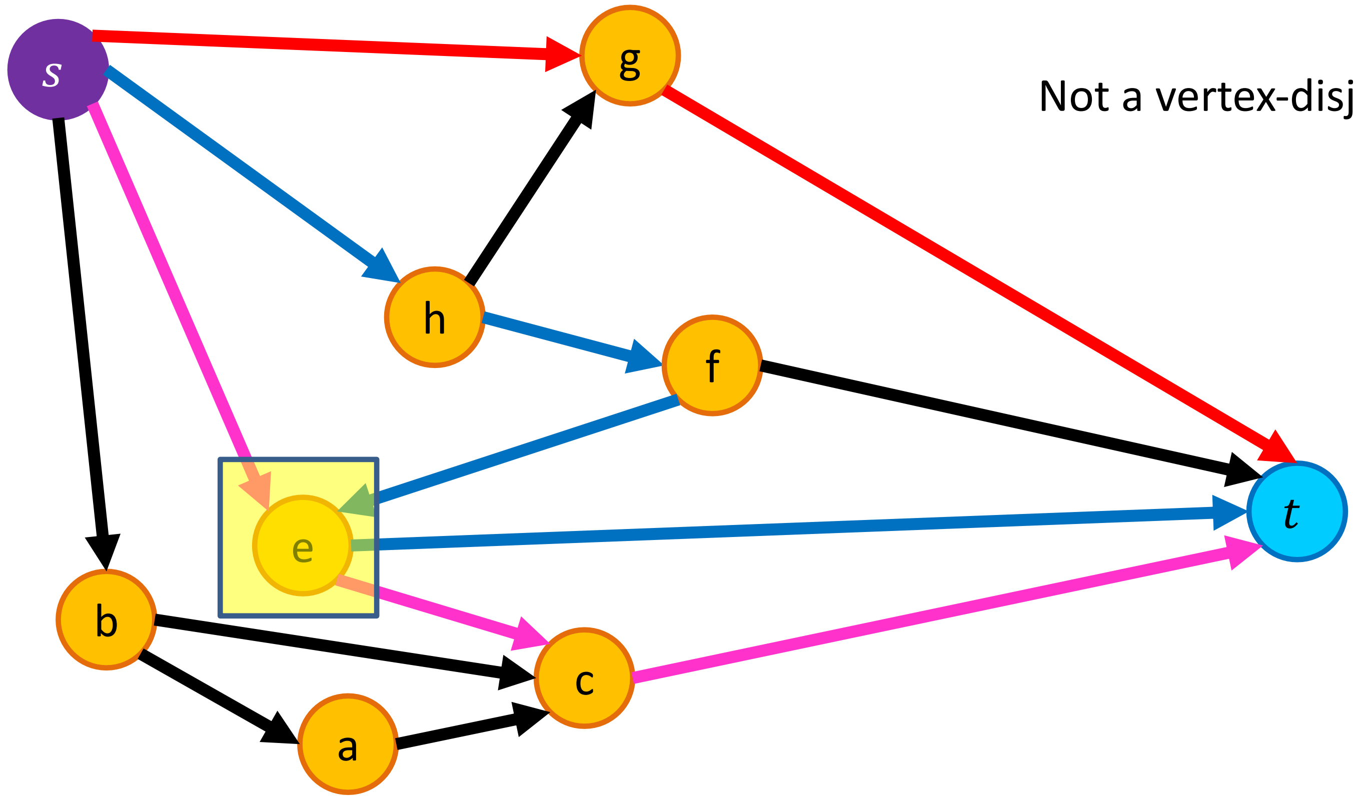
Vertex-Disjoint Paths

Given a graph $G = (V, E)$, a start node s and a destination node t, give the maximum number of paths from s to t which share no vertices



Vertex-Disjoint Paths

Given a graph $G = (V, E)$, a start node s and a destination node t, give the maximum number of paths from s to t which share no vertices



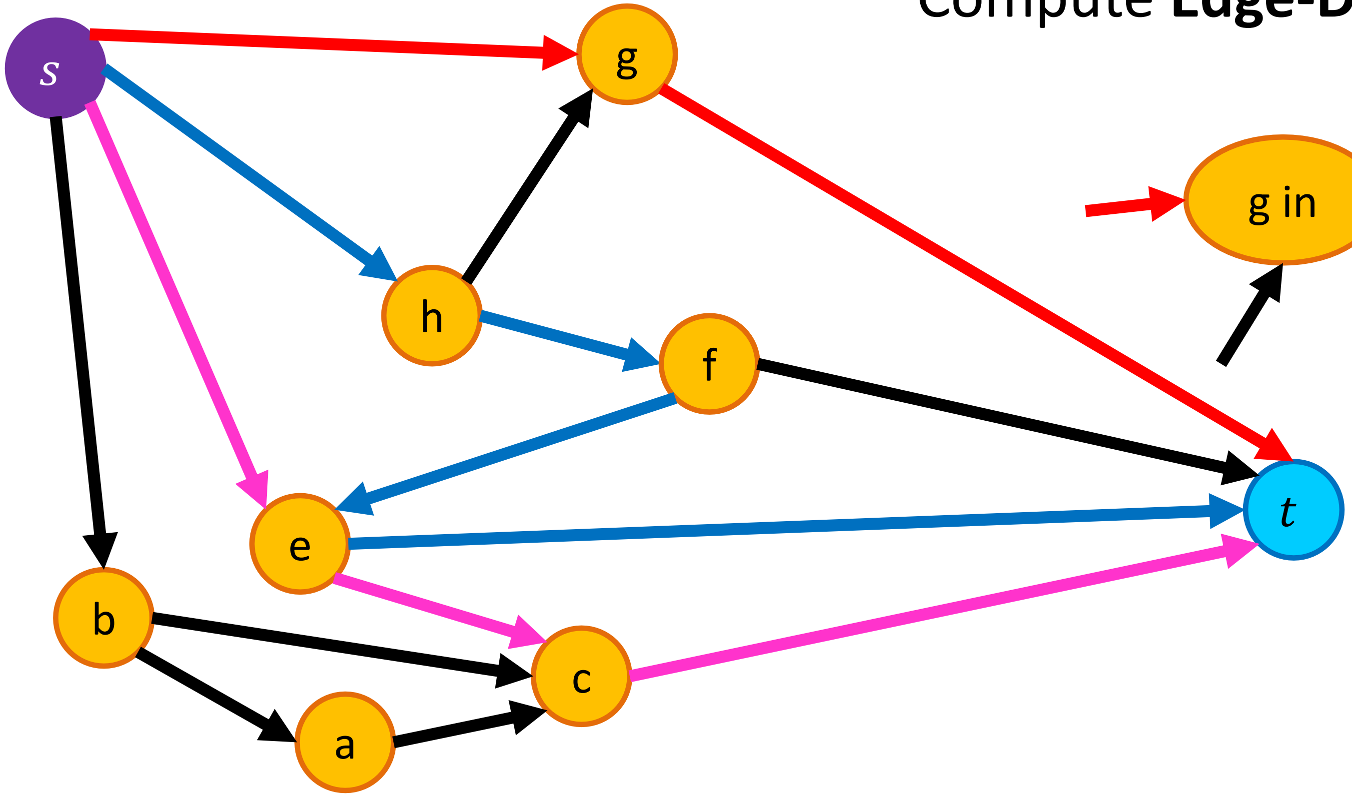
Not a vertex-disjoint path!

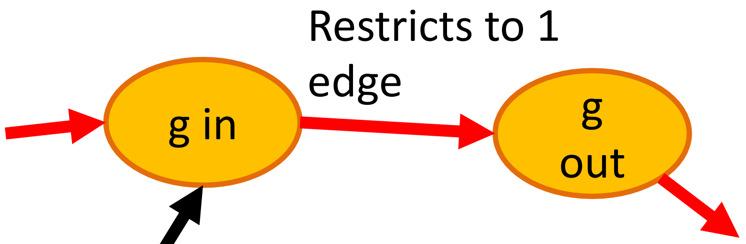
Vertex-Disjoint Paths Algorithm

Idea: Convert an instance of the vertex-disjoint paths problem into an instance of edge-disjoint paths

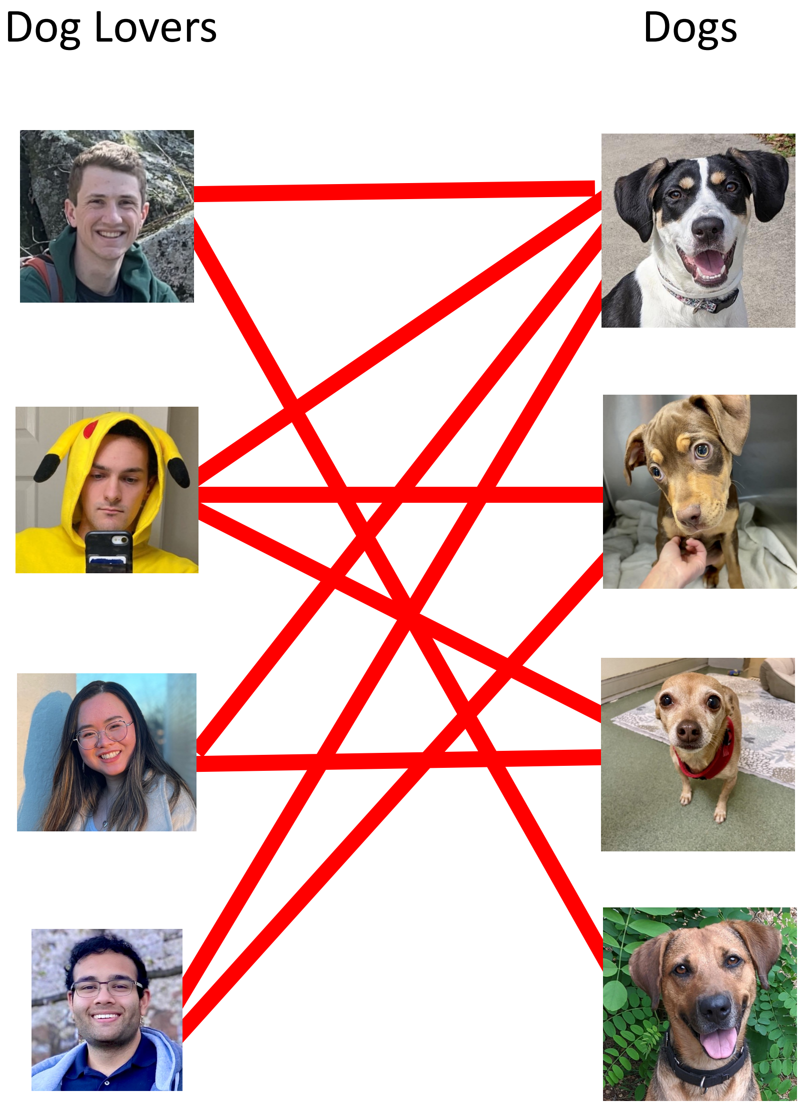
Make two copies of each node, one connected to incoming edges, the other to outgoing edges

Compute **Edge-Disjoint Paths** on new graph





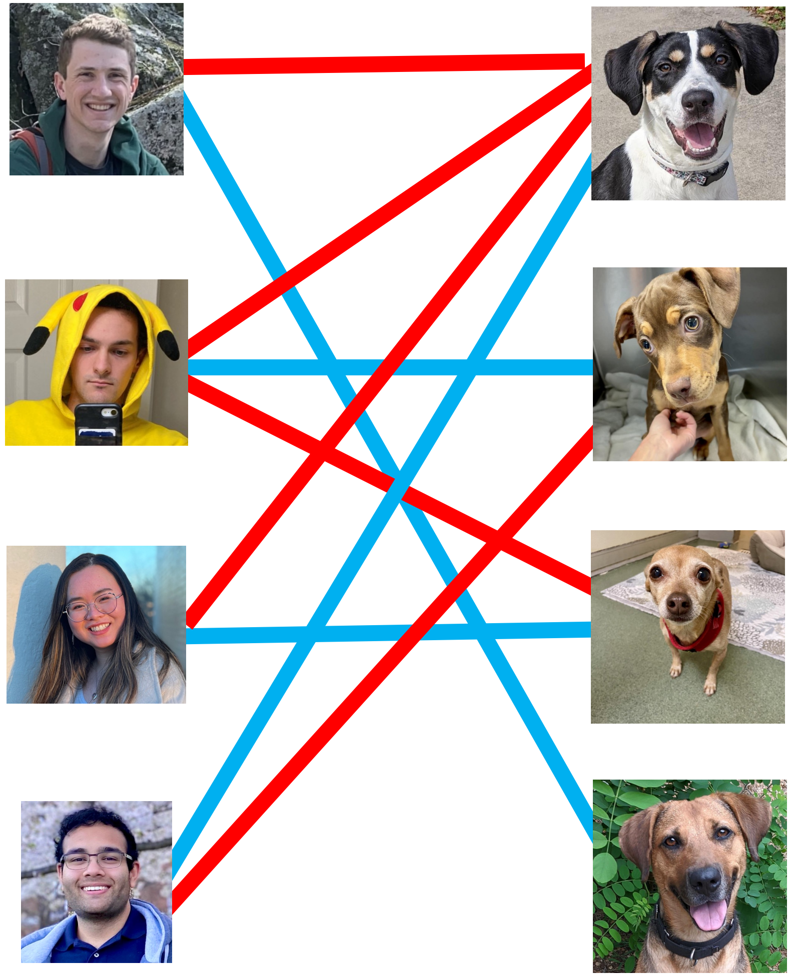
Maximum Bipartite Matching



Maximum Bipartite Matching

Maximum Bipartite Matching

Dog Lovers | Dogs



Maximum Bipartite Matching

Given a graph $G = (L, R, E)$

a set of left nodes, right nodes, and edges between left and right

Find the largest set of edges $M \subseteq E$ such that each node $u \in L$ or $v \in R$ is incident to at most one edge.

Maximum Bipartite Matching Using Max Flow

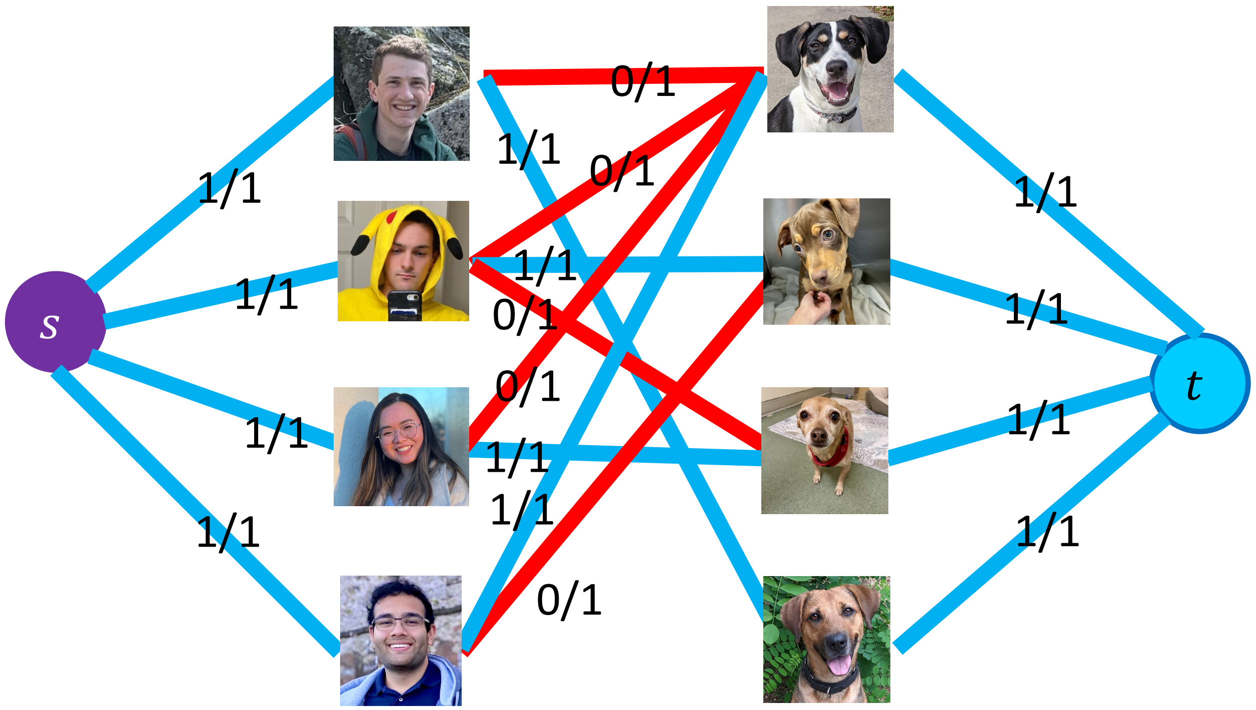
Make $G = (L, R, E)$ a flow network $G' = (V', E')$ by:

- Adding in a source and sink to the set of nodes:
  - $V' = L \cup R \cup \{s, t\}$
- Adding an edge from source to L and from R to sink:
  - $E' = E \cup \{u \in L \mid (s, u)\} \cup \{v \in r \mid (v, t)\}$
- Make each edge capacity 1:
  - $\forall e \in E', c(e) = 1$

Maximum Bipartite Matching Using Max Flow

$\Theta(E \cdot V)$

1. Make G into G' $\Theta(L + R)$
2. Compute Max Flow on G' $\Theta(E \cdot V)$ $|f| \leq L$
3. Return M as all "middle" edges with flow 1 $\Theta(L + R)$



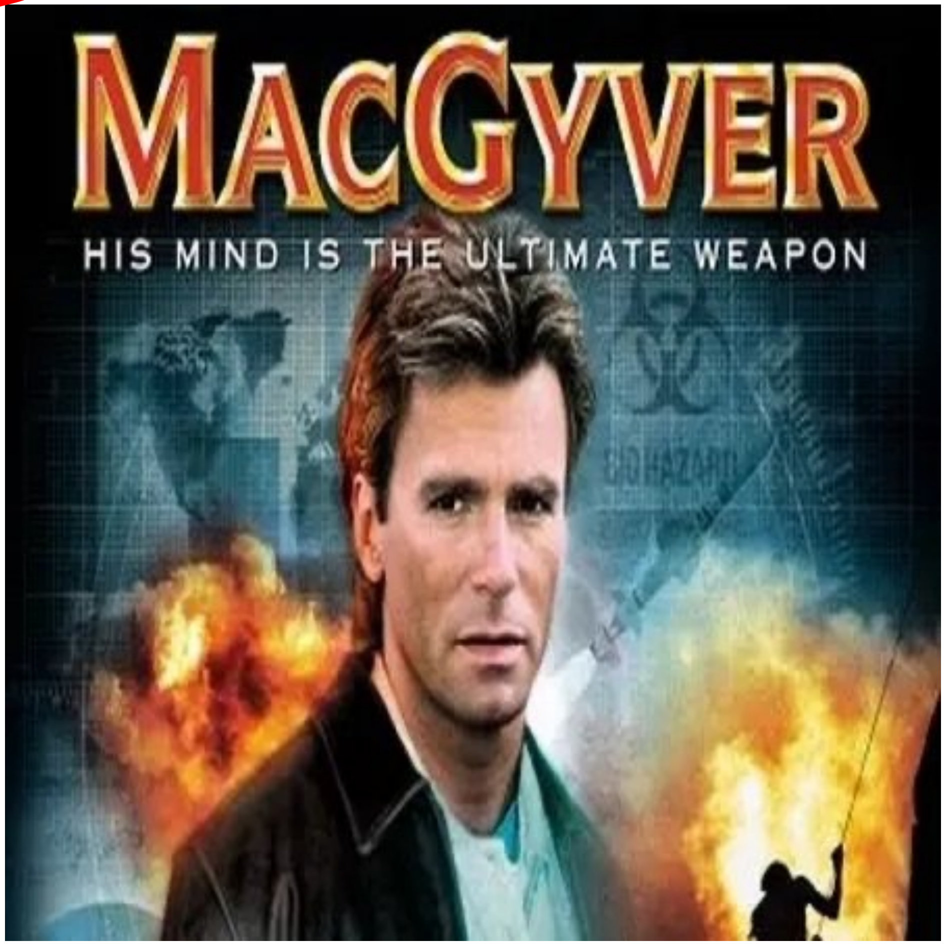
Reductions

- Algorithm technique of supreme ultimate power
- Convert instance of problem A to an instance of Problem B
- Convert solution of problem B back to a solution of problem A

Reductions

Shows how two different problems relate to each other

MOVIE TIME!



MacGyver's Reduction

Problem we don't know how to solve

A

Opening a door

Solution for $\boldsymbol{A}$

Keg cannon
battering ram

Aim duct at door,
insert keg

Put fire under the Keg

Reduction

Problem we do know how to solve

B

Lighting a fire

How?

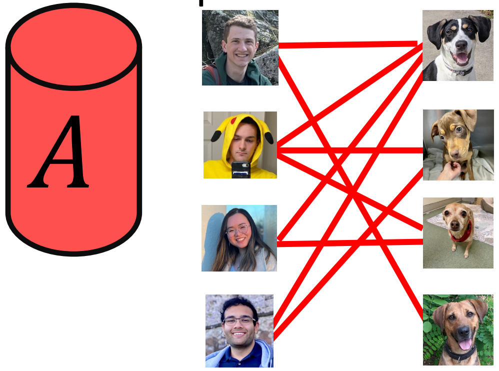
Solution for $\boldsymbol{B}$

Alcohol, wood,
matches

Bipartite Matching Reduction

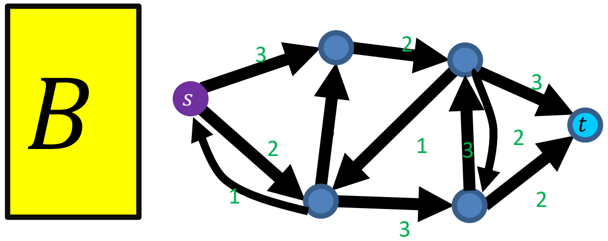
Problem we don't know how to solve

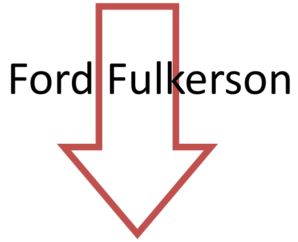
Bipartite Matching

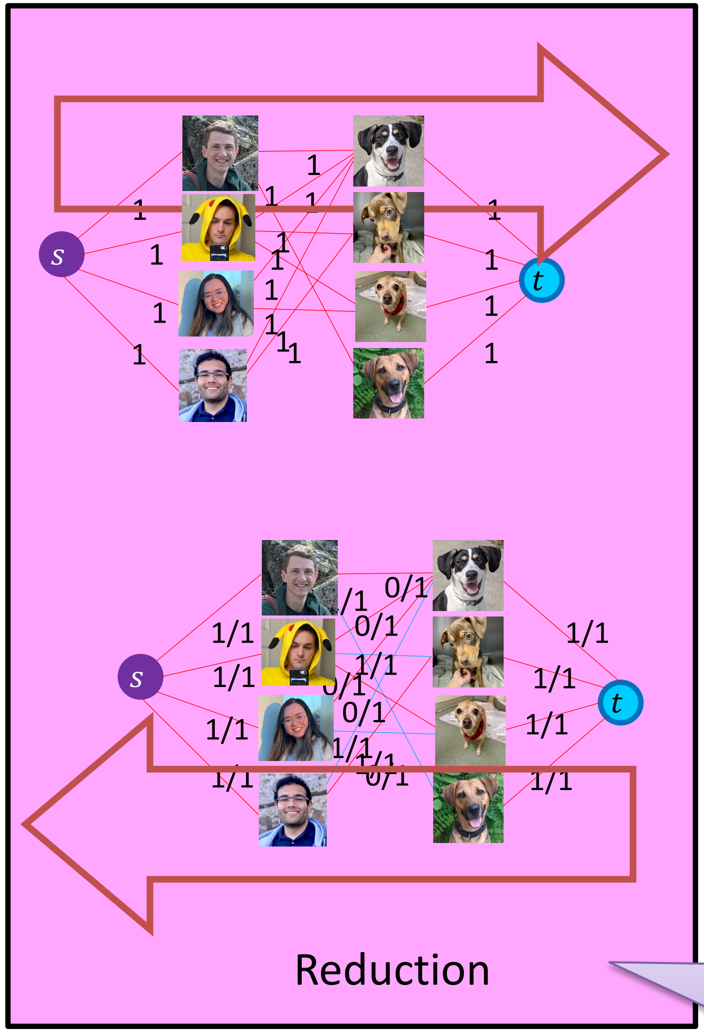


Problem we do know how to solve

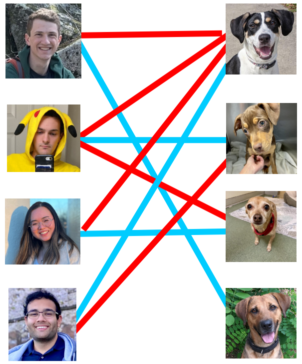
Max Flow



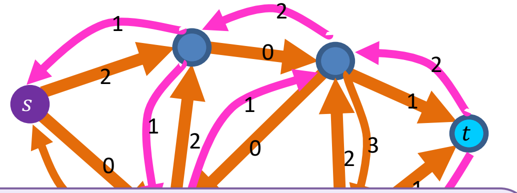




Solution for $\boldsymbol{A}$

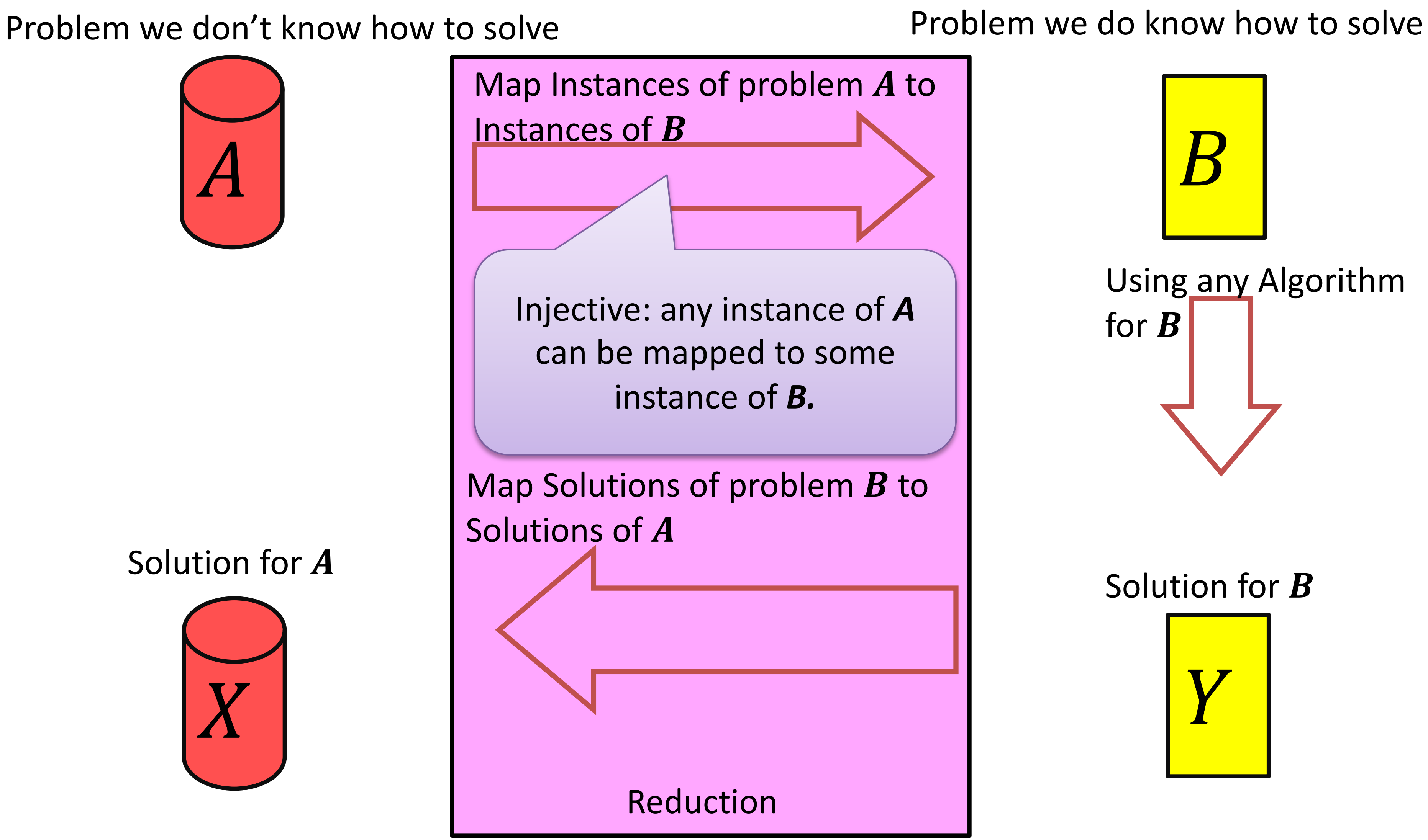


Solution for $\boldsymbol{B}$



Must show (prove):
1) how to make construction
2) Why it works

In General: Reduction



Worst-case lower-bound Proofs

Opening a door

A

Problem **A**

reduces to

B

Problem **B**

Lighting a fire

Alcohol, wood, matches

Y

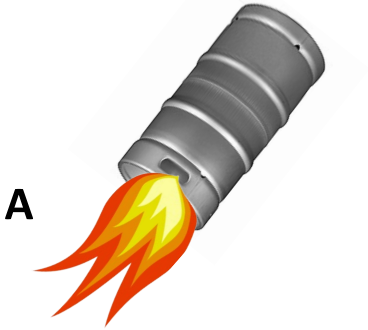
Algorithm for **B**

can be used to make

X

Algorithm for **A**

Keg cannon battering ram



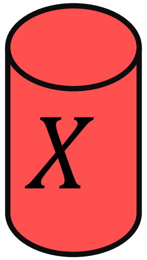
A is not a harder problem than B

$$A \leq B$$

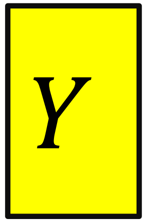
The name "reduces" is confusing: it is in the *opposite* direction of the making

Proof of Lower Bound by Reduction

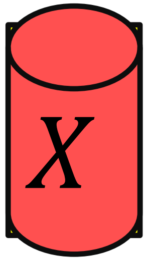
<u>To Show: Y is slow</u>



1. We know X is slow (by a proof)
(e.g., X = some way to open the door)



2. Assume Y is quick [toward contradiction]
(Y = some way to light a fire)



3. Show how to use Y to perform X quickly

4. X is slow, but Y could be used to perform X quickly
conclusion: Y must not actually be quick