CS 4102: Algorithms – Unit C

Dynamic Programming

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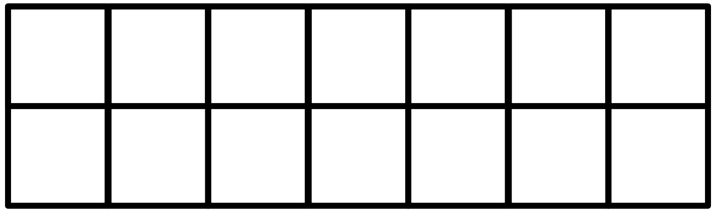
Spring 2022

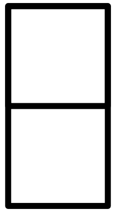
Warm Up

How many ways are there to tile a $2 \times n$ board with dominoes?

How many ways to tile this:



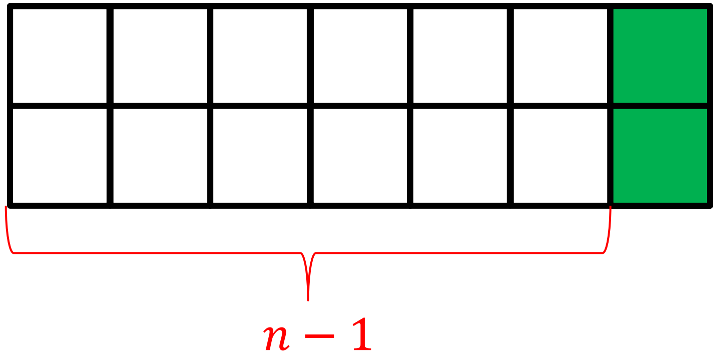
With these?



Warm Up

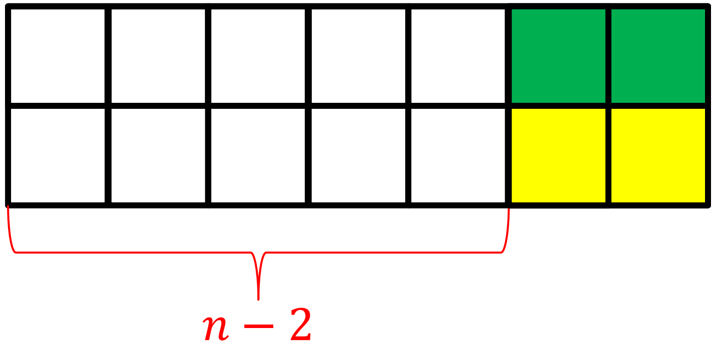
How many ways are there to tile a $2\times n$ board with dominoes?

Two ways to fill the final column:



$$Tile(n) = Tile(n-1) + Tile(n-2)$$

$$Tile(0) = Tile(1) = 1$$



Today's Keywords

- Maximum Sum Continuous Subarray
- Domino Tiling
- Dynamic Programming
- Log Cutting

CLRS Readings

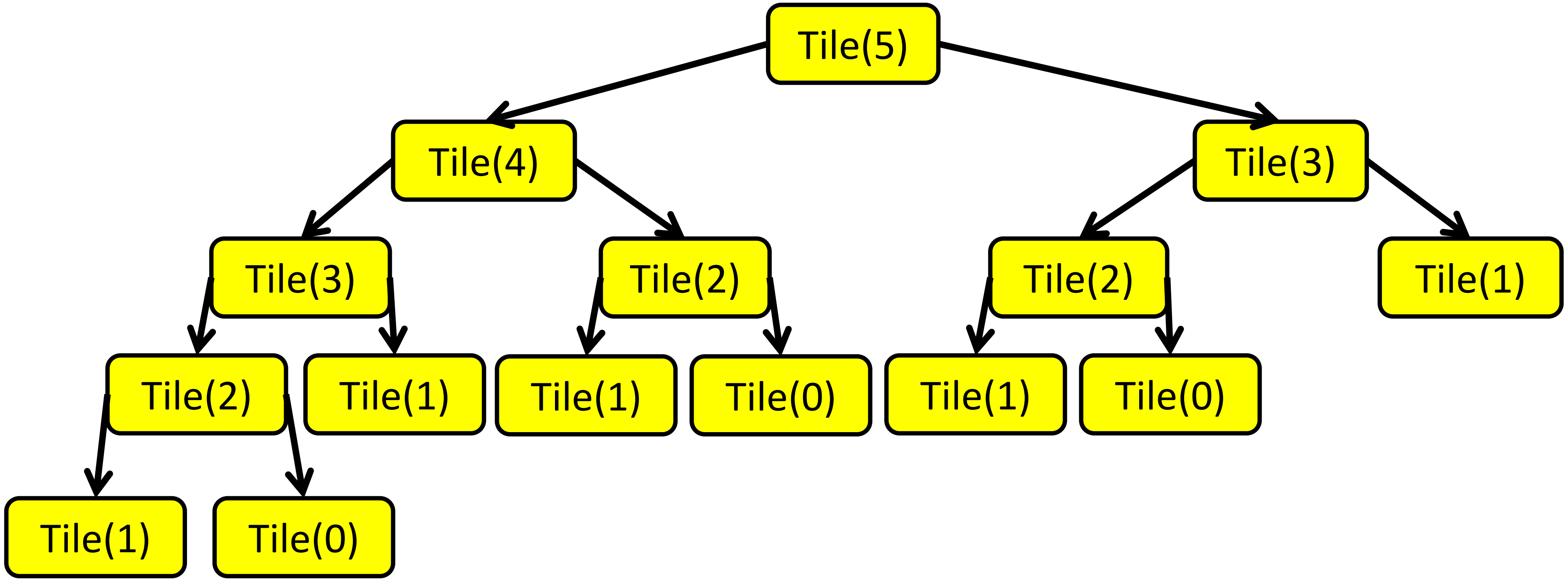
- Chapter 15
  - Section 15.1, Log/Rod cutting, optimal substructure property
    - Note: r_i in book is called Cut() or C[] in our slides. We use their example.
  - Section 15.3, More on elements of DP, including optimal substructure property
  - Section 15.2, matrix-chain multiplication (later example)
  - Section 15.4, longest common subsequence (even later example)

How to compute $Tile(n)$?

```
Tile(n):
    if n < 2:
        return 1
    return Tile(n-1)+Tile(n-2)
```

Problem?

Recursion Tree



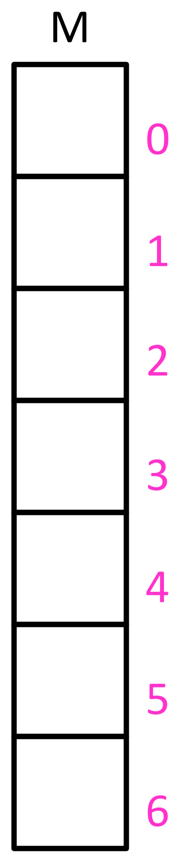
Many redundant calls!

Run time: $\Omega(2^n)$

Better way: Use Memory!

Computing $Tile(n)$ with Memory

```
Initialize Memory M
Tile(n):
        if n < 2:
                return 1
        if M[n] is filled:
                return M[n]
        M[n] = Tile(n-1)+Tile(n-2)
        return M[n]
```



Technique: “memoization” (note no “r”)

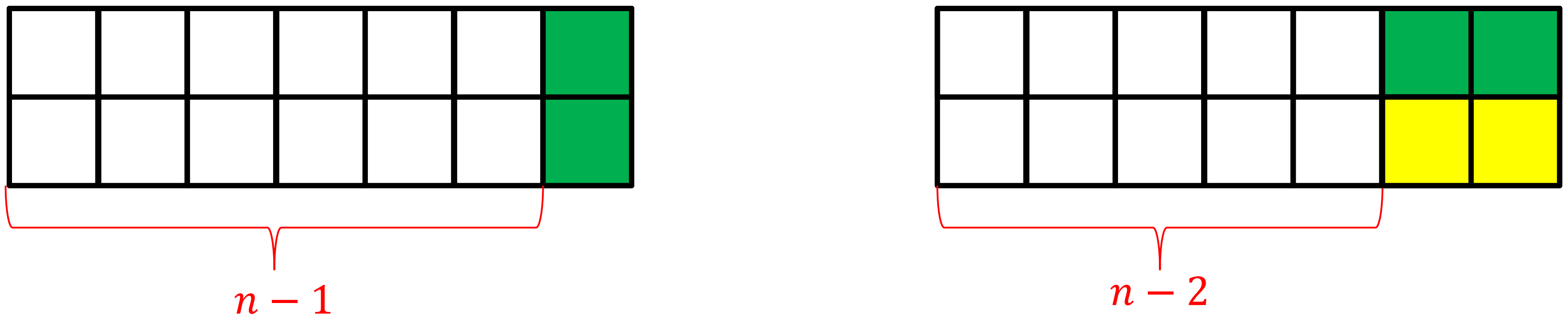
Computing $Tile(n)$ with Memory - "Top Down"

```
Initialize Memory M
Tile(n):
    if n < 2:
        return 1
    if M[n] is filled:
        return M[n]
    M[n] = Tile(n-1)+Tile(n-2)
    return M[n]
```

M	
1	0
1	1
2	2
3	3
5	4
8	5
13	6

Dynamic Programming

- Requires Optimal Substructure
  – Solution to larger problem contains the (optimal) solutions to smaller ones
- Idea:
  1. Identify recursive structure of the problem
     - What is the "last thing" done?



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  1. Identify the recursive structure of the problem
     - What is the “last thing” done?
  2. Save the solution to each subproblem in memory

Generic Divide and Conquer Solution

```
def myDCalgo(problem):

    if baseCase(problem):
        solution = solve(problem)

        return solution
    for subproblem of problem:    # After dividing
        subsolutions.append(myDCalgo(subproblem))
    solution = Combine(subsolutions)

    return solution
```

Generic Top-Down Dynamic Programming Soln

```
mem = {}
def myDPalgo(problem):
        if mem[problem] not blank:
                return mem[problem]
        if baseCase(problem):
                solution = solve(problem)
                mem[problem] = solution
                return solution
        for subproblem of problem:
                subsolutions.append(myDPalgo(subproblem))
        solution = OptimalSubstructure(subsolutions)
        mem[problem] = solution
        return solution
```

Computing $Tile(n)$ with Memory - “Top Down”

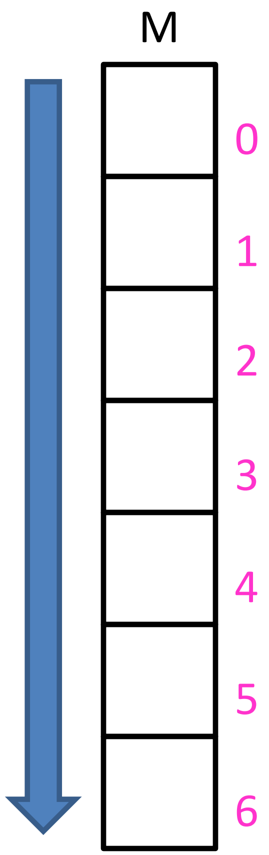
```
Initialize Memory M
Tile(n):
        if n < 2:
                return 1
        if M[n] is filled:
                return M[n]
        M[n] = Tile(n-1)+Tile(n-2)
        return M[n]
```

M	
1	0
1	1
2	2
3	3
5	4
8	5
13	6

Recursive calls happen in a predictable order

Better $Tile(n)$ with Memory - "Bottom Up"

```
Tile(n):
    Initialize Memory M
    M[0] = 1
    M[1] = 1
    for i = 2 to n:
        M[i] = M[i-1] + M[i-2]
    return M[n]
```



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More on Optimal Substructure Property

- Detailed discussion on CLRS p. 379
  - If A is an optimal solution to a problem, then the components of A are optimal solutions to subproblems
- Examples (we'll see these come up later):
  - True for coin-changing
  - True for single-source shortest path
  - True for knapsack problem

Log Cutting

Given a log of length n
A list (of length n) of prices P ($P[i]$ is the price of a cut of size i)
Find the best way to cut the log

Price:	1	5	8	9	10	17	17	20	24	30
Length:	1	2	3	4	5	6	7	8	9	10

Select a list of lengths $\ell_1, \ldots, \ell_k$ such that:
$\sum \ell_i = n$
to maximize $\sum P[\ell_i]$

Brute Force: $O(2^n)$

Greedy won't work

- Greedy algorithms (next unit) build a solution by picking the best option "right now"
  - Select the most profitable cut first

Price:	1	18	24	36	50	50
Length:	1	2	3	4	5	6

Greedy: Lengths: 5, 1
Profit: 51

Better: Lengths: 2, 4
Profit: 54

Greedy won't work

- Greedy algorithms (next unit) build a solution by picking the best option "right now"
  - Select the "most bang for your buck"
    - (best price / length ratio)

Price:	1	18	24	36	50	50
Length:	1	2	3	4	5	6

Greedy: Lengths: 5, 1
Profit: 51

Better: Lengths: 2, 4
Profit: 54

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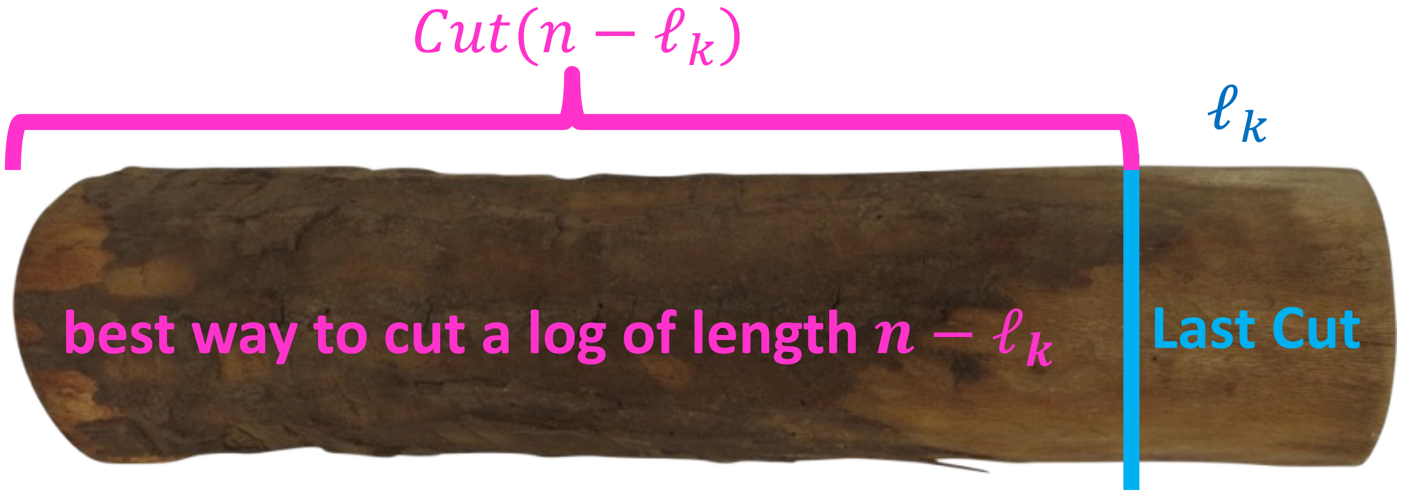
1. Identify Recursive Structure

$P[i]$ = value of a cut of length i

$Cut(n)$ = value of best way to cut a log of length n

$$Cut(n) = \max \begin{cases} Cut(n-1) + P[1] \\ Cut(n-2) + P[2] \\ \dots \\ Cut(0) + P[n] \end{cases}$$



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3. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$$Cut(0) = 0$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

0

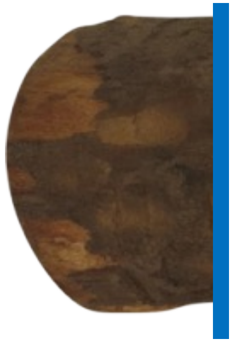
3. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$$Cut(1) = Cut(0) + P[1]$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

1



3. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$$Cut(2) = \max \begin{cases} Cut(1) + P[1] \\ Cut(0) + P[2] \end{cases}$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

3. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$$Cut(3) = \max \begin{cases} Cut(2) + P[1] \\ Cut(1) + P[2] \\ Cut(0) + P[3] \end{cases}$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

3

3. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$$Cut(4) = \max \begin{cases} Cut(3) + P[1] \\ Cut(2) + P[2] \\ Cut(1) + P[3] \\ Cut(0) + P[4] \end{cases}$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

4

Log Cutting Pseudocode

```
Initialize Memory C
Cut(n):
    C[0] = 0
    for i=1 to n:  // log size
        best = 0
        for j = 1 to i: // last cut
            best = max(best, C[i-j] + P[j])
        C[i] = best
    return C[n]
```

Run Time: $O(n^2)$

How to find the cuts?

- This procedure told us the profit, but not the cuts themselves
- Idea: remember the choice that you made, then backtrack

Remember the choice made

```
Initialize Memory C, Choices
Cut(n):
    C[0] = 0
    for i=1 to n:
        best = 0
        for j = 1 to i:
            if best < C[i-j] + P[j]:
                best = C[i-j] + P[j]
                Choices[i]=j
        C[i] = best
    return C[n]
```

Gives the size of the last cut

Reconstruct the Cuts

- Backtrack through the choices

Choices:	0	1	1	2	4	3	4	1	2	4	3
Length:	0	1	2	3	4	5	6	7	8	9	10

Example to demo Choices[] only. Profit of 20 is not optimal!

Backtracking Pseudocode

```
i = n
while i > 0:
        print Choices[i]
        i = i – Choices[i]
```

Our Example: Getting Optimal Solution

i	0	1	2	3	4	5	6	7	8	9	10
C[i]	0	1	5	8	10	13	17	18	22	25	30
Choice[i]	0	1	2	3	2	2	6	1	2	3	10

- If n were 5
  - Best score is 13
  - Cut at Choice[n]=2, then cut at Choice[n-Choice[n]]= Choice[5-2]= Choice[3]=3
- If n were 7
  - Best score is 18
  - Cut at 1, then cut at 6

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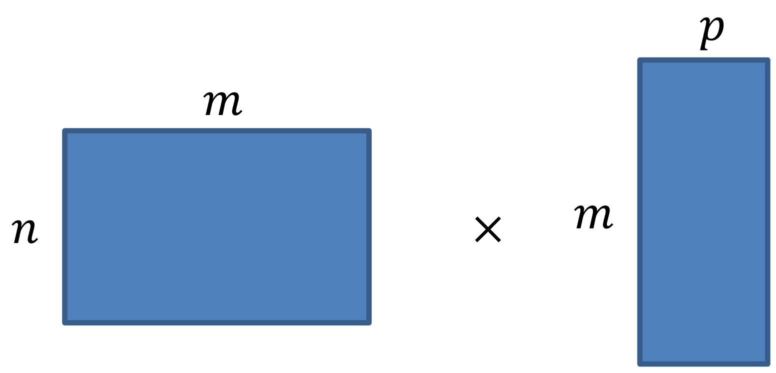
Mental Stretch

How many arithmetic operations are required to multiply a $n \times m$ Matrix with a $m \times p$ Matrix?

(don't overthink this)



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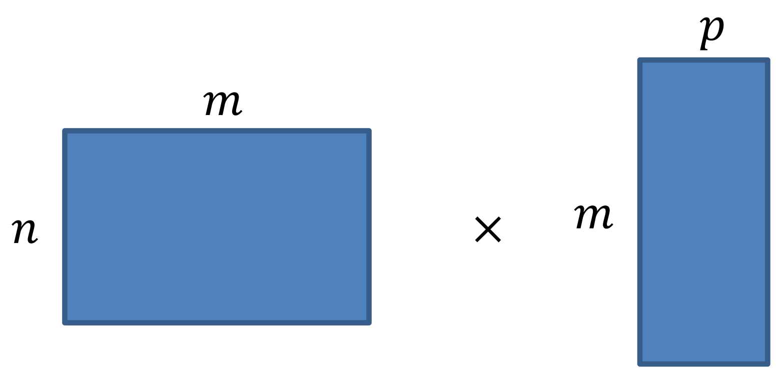
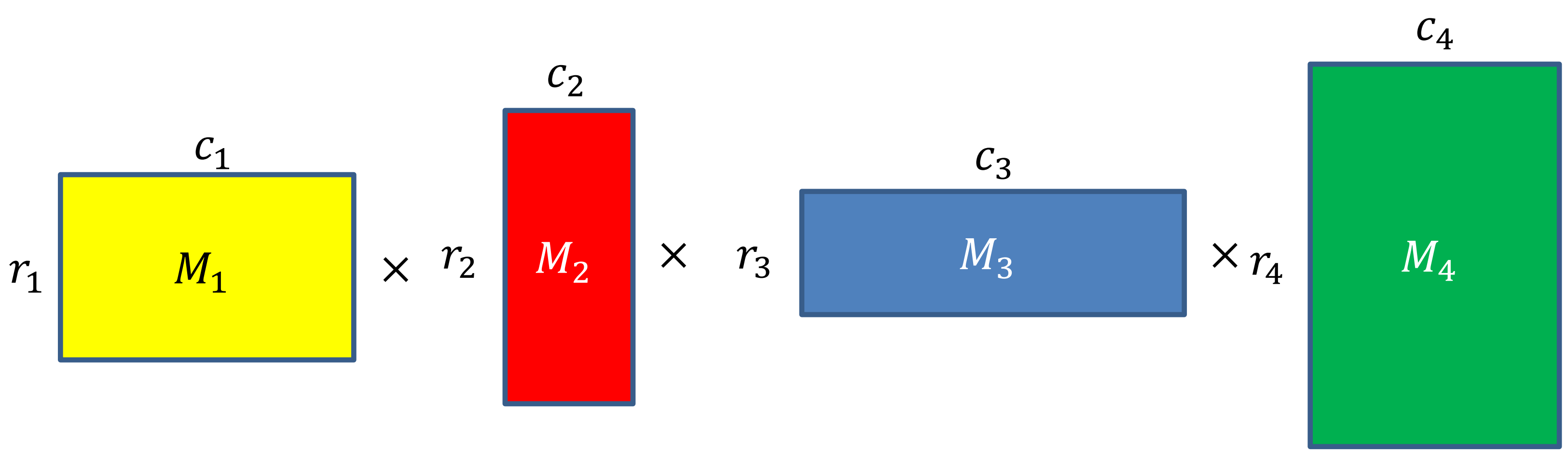


- m multiplications and additions per element
- $n \cdot p$ elements to compute
- Total cost: $m \cdot n \cdot p$

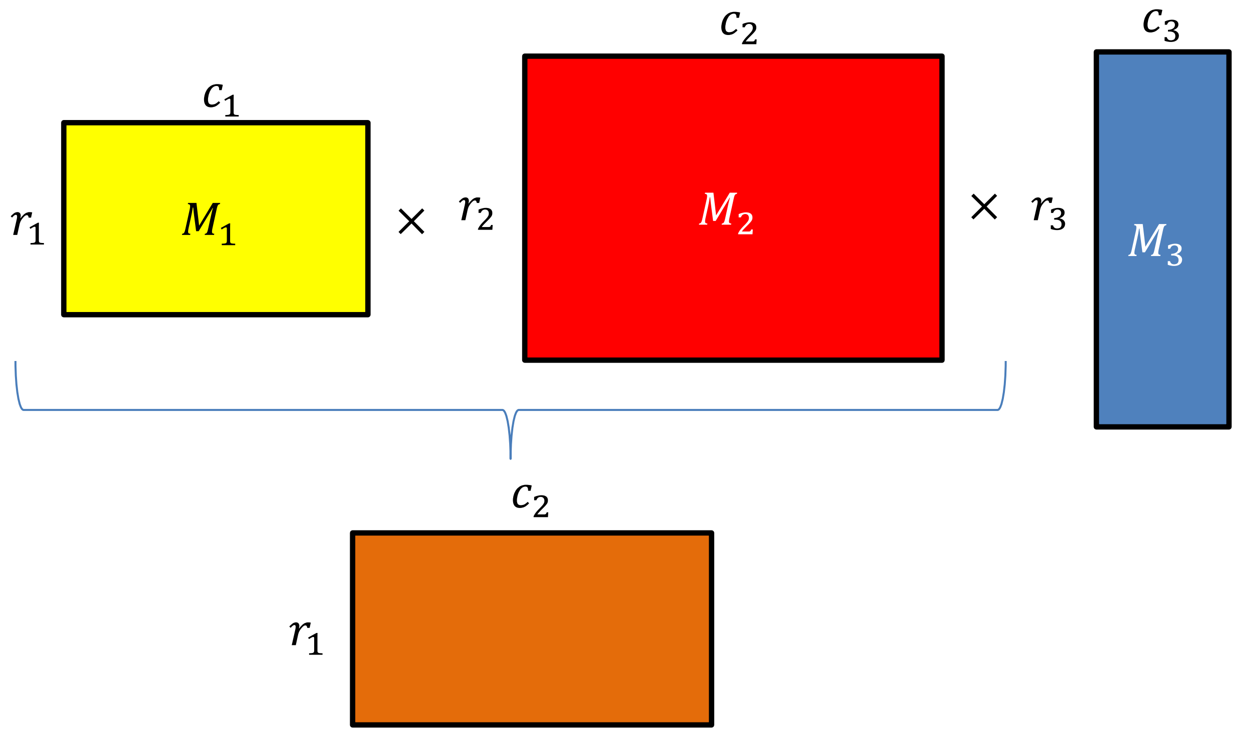
Matrix Chaining

- Given a sequence of Matrices $(M_1, \dots, M_n)$, what is the most efficient way to multiply them?



Order Matters!

$c_1 = r_2$
$c_2 = r_3$



- $(M_1 \times M_2) \times M_3$
  - uses $(c_1 \cdot r_1 \cdot c_2) + c_2 \cdot r_1 \cdot c_3$ operations

Order Matters!

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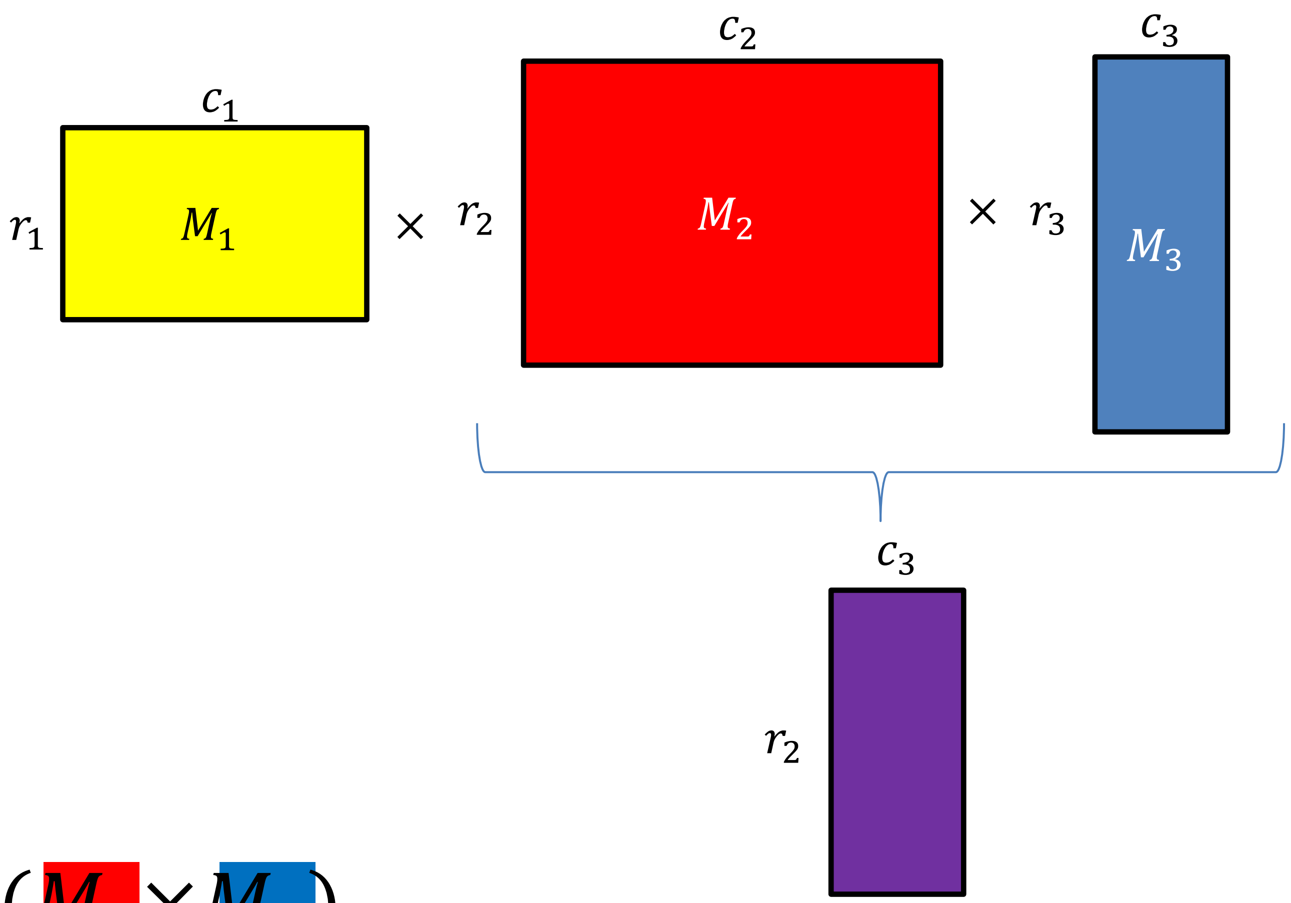

- $M_1 \times (M_2 \times M_3)$
  - uses $\mathrm{c}_1 \cdot \mathrm{r}_1 \cdot c_3 + (\mathrm{c}_2 \cdot r_2 \cdot c_3)$ operations

Order Matters!

$c_1 = r_2$
$c_2 = r_3$

- $(M_1 \times M_2) \times M_3$
  - uses $(c_1 \cdot r_1 \cdot c_2) + \mathrm{c}_2 \cdot r_1 \cdot c_3$ operations
  - $(10 \cdot 7 \cdot 20) + 20 \cdot 7 \cdot 8 = 2520$
- $M_1 \times (M_2 \times M_3)$
  - uses $c_1 \cdot r_1 \cdot c_3 + (\mathrm{c}_2 \cdot r_2 \cdot c_3)$ operations
  - $10 \cdot 7 \cdot 8 + (20 \cdot 10 \cdot 8) = 2160$

$M_1 = 7 \times 10$
$M_2 = 10 \times 20$
$M_3 = 20 \times 8$

$c_1 = 10$
$c_2 = 20$
$c_3 = 8$
$r_1 = 7$
$r_2 = 10$
$r_3 = 20$

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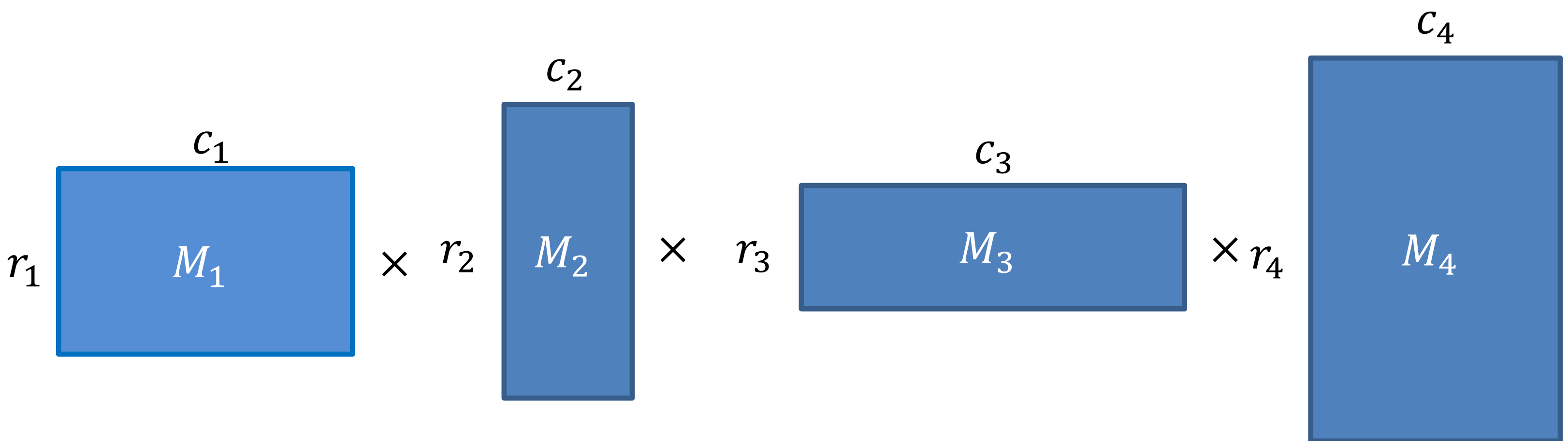
1. Identify the Recursive Structure of the Problem

$Best(1, n)$ = cheapest way to multiply together M_1 through M_n



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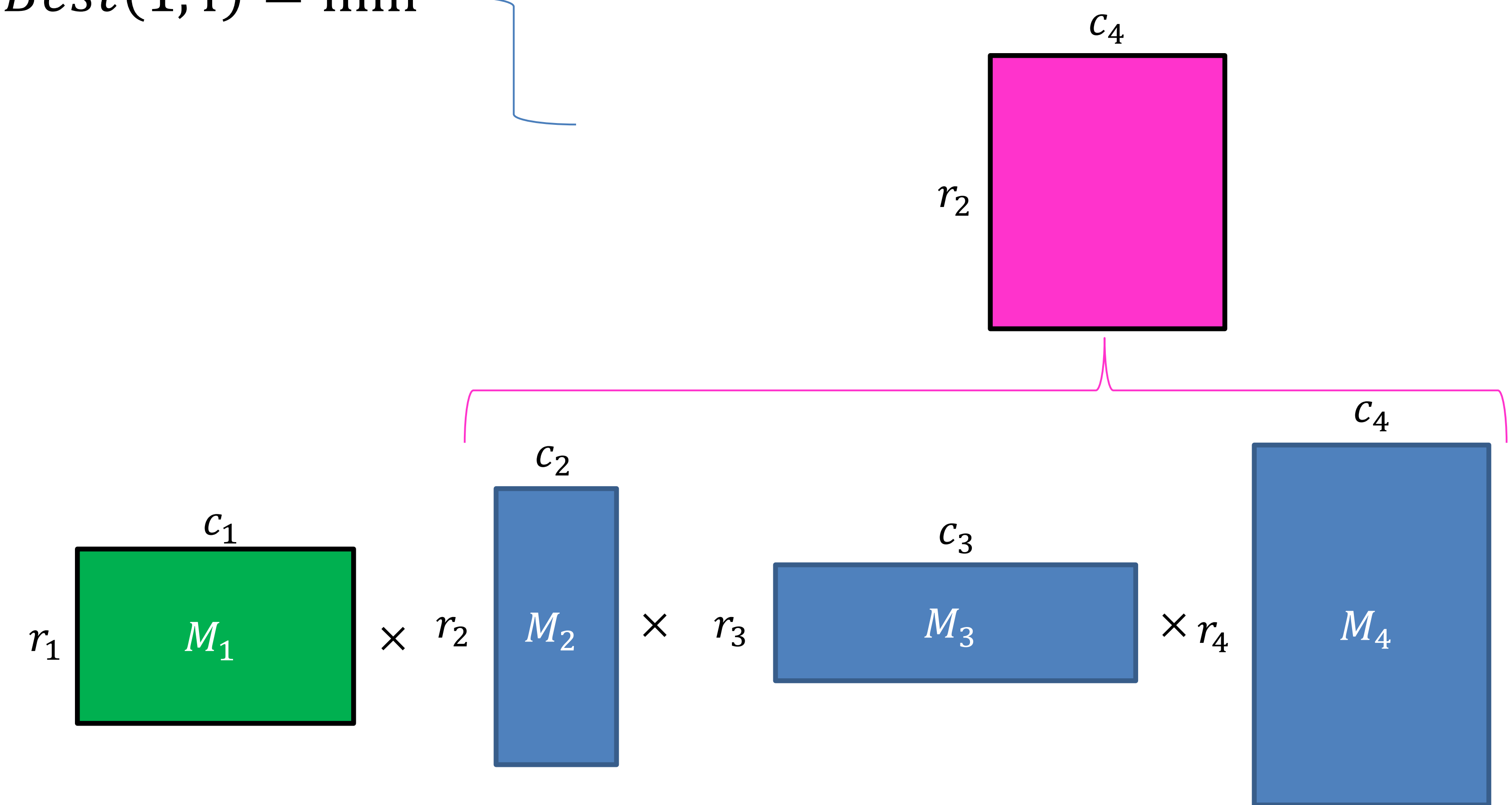
$Best(1, n)$ = cheapest way to multiply together M_1 through M_n

$$Best(1,4) = \min \begin{cases} Best(2,4) + r_1 r_2 c_4 \\ \\ \end{cases}$$



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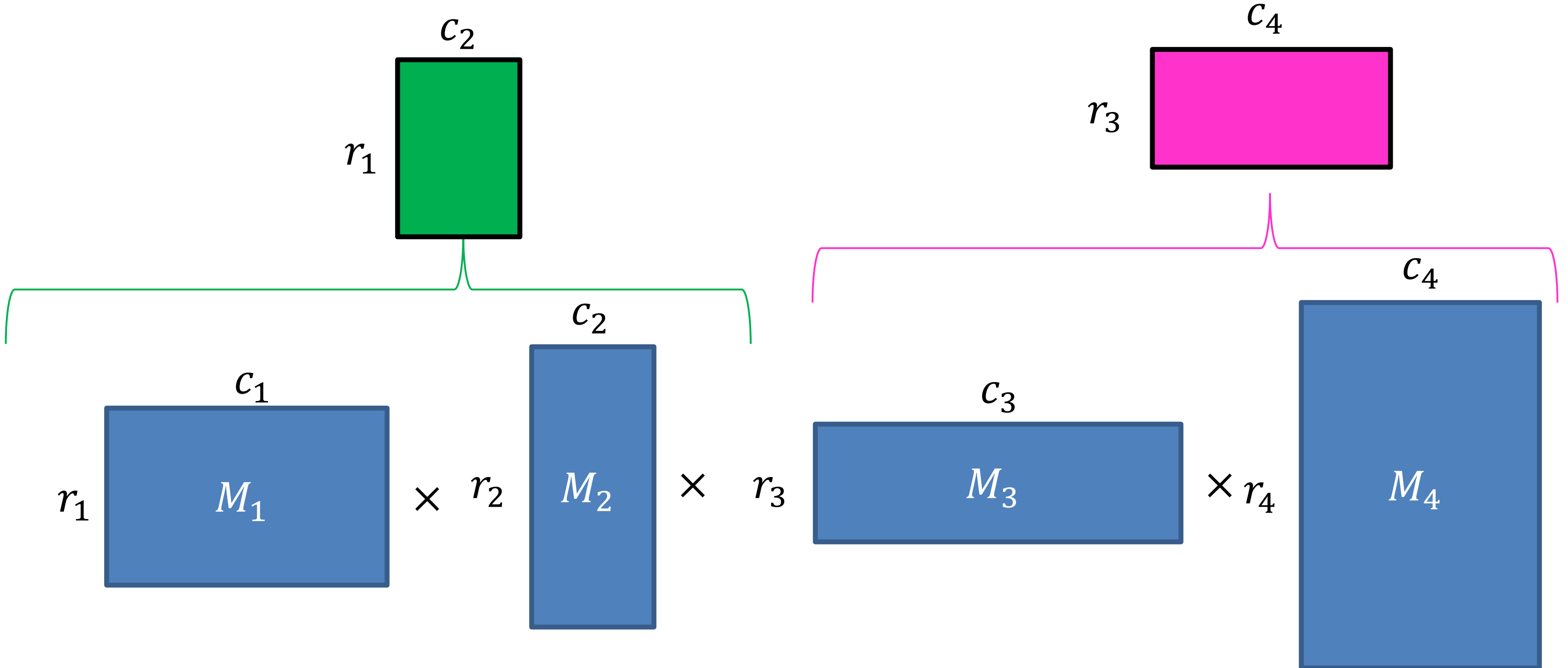

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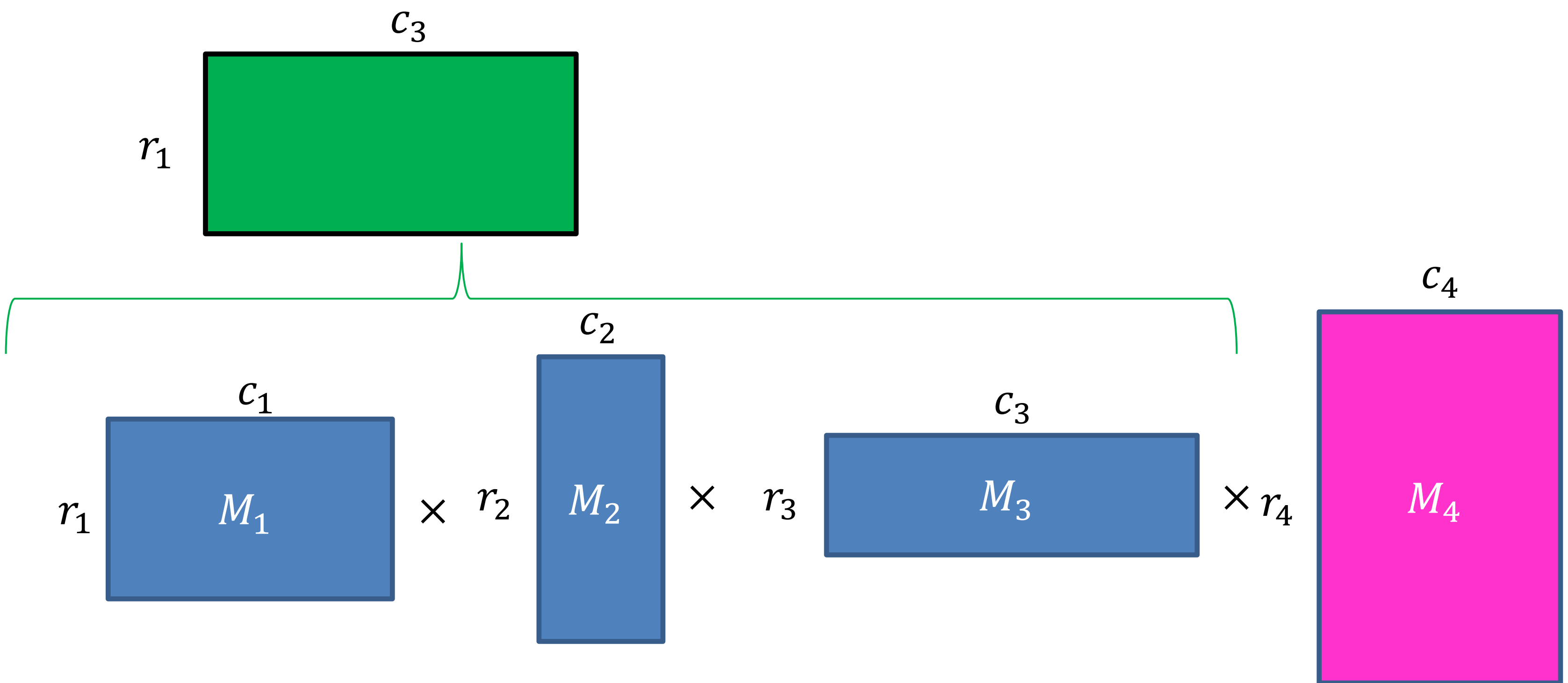

1. Identify the Recursive Structure of the Problem

- In general:

$Best(i,j) =$ cheapest way to multiply together M_i through M_j

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$$Best(1,n) = \min \begin{cases} Best(2,n) + r_1 r_2 c_n \\ Best(1,2) + Best(3,n) + r_1 r_3 c_n \\ Best(1,3) + Best(4,n) + r_1 r_4 c_n \\ Best(1,4) + Best(5,n) + r_1 r_5 c_n \\ \quad \ldots \\ Best(1,n-1) + r_1 r_n c_n \end{cases}$$

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2. Save Subsolutions in Memory

- In general:

$Best(i,j)$ = cheapest way to multiply together M_i through M_j

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

Read from M[n] if present

Save to M[n]

$$Best(1,n) = \min \begin{cases} Best(2,n) + r_1 r_2 c_n \\ Best(1,2) + Best(3,n) + r_1 r_3 c_n \\ Best(1,3) + Best(4,n) + r_1 r_4 c_n \\ Best(1,4) + Best(5,n) + r_1 r_5 c_n \\ \quad \ldots \\ Best(1,n-1) + r_1 r_n c_n \end{cases}$$

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3. Select a good order for solving subproblems

- In general:

$$Best(i,j) = \text{cheapest way to multiply together } M_i \text{ through } M_j$$

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

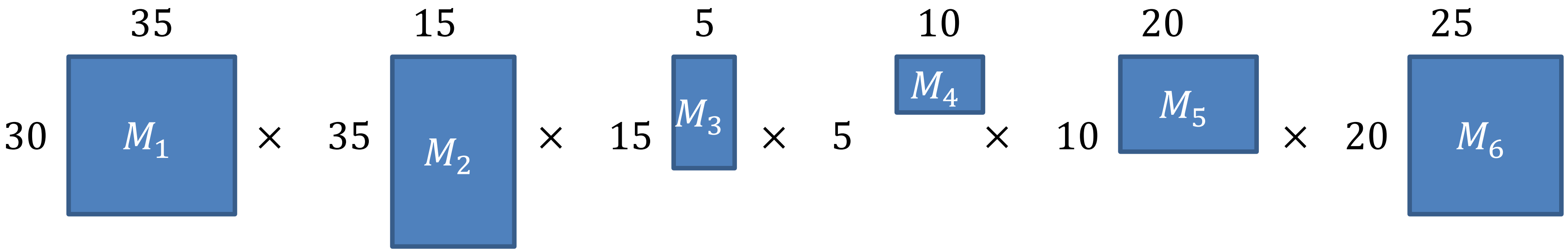
$$Best(i,i) = 0$$

Read from M[n] if present

Save to M[n]

$$Best(1,n) = \min \begin{cases} Best(2,n) + r_1 r_2 c_n \\ Best(1,2) + Best(3,n) + r_1 r_3 c_n \\ Best(1,3) + Best(4,n) + r_1 r_4 c_n \\ Best(1,4) + Best(5,n) + r_1 r_5 c_n \\ \quad \ldots \\ Best(1,n-1) + r_1 r_n c_n \end{cases}$$

3. Select a good order for solving subproblems



$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$i =$
0						1
	0					2
		0				3
			0			4
				0		5
					0	6

3. Select a good order for solving subproblems

$$\underset{30\times 35}{M_1} \times \underset{35\times 15}{M_2} \times \underset{15\times 5}{M_3} \times \underset{5\times 10}{M_4} \times \underset{10\times 20}{M_5} \times \underset{20\times 25}{M_6}$$

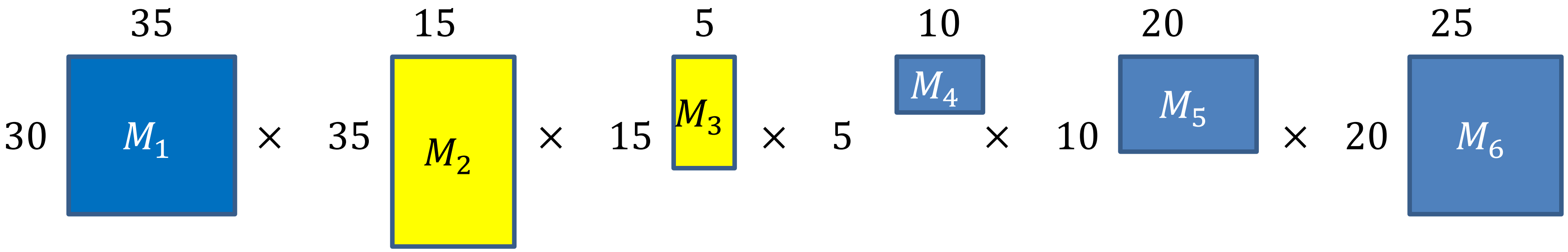
$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$i =$
0	15750					1
	0					2
		0				3
			0			4
				0		5
					0	6

$$Best(1,2) = \min\left\{ Best(1,1) + Best(2,2) + r_1 r_2 c_2 \right.$$

3. Select a good order for solving subproblems



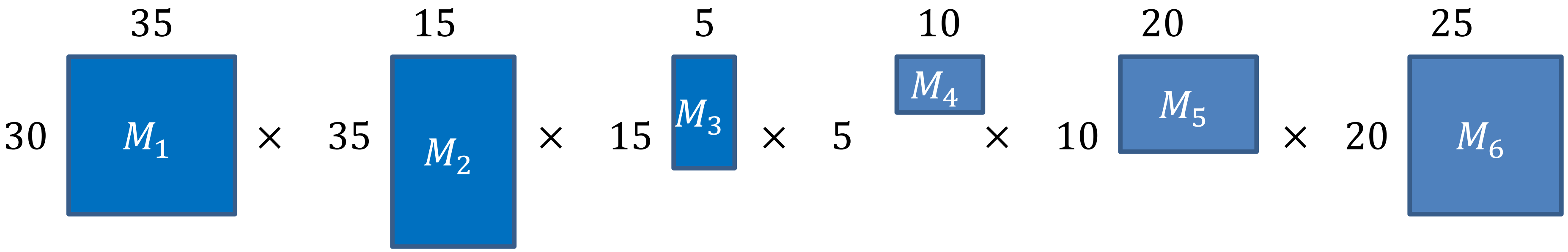
$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

i \ $j =$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0			
4				0		
5					0	
6						0

$$Best(2,3) = \min\left\{ Best(2,2) + Best(3,3) + r_2 r_3 c_3 \right.$$

3. Select a good order for solving subproblems

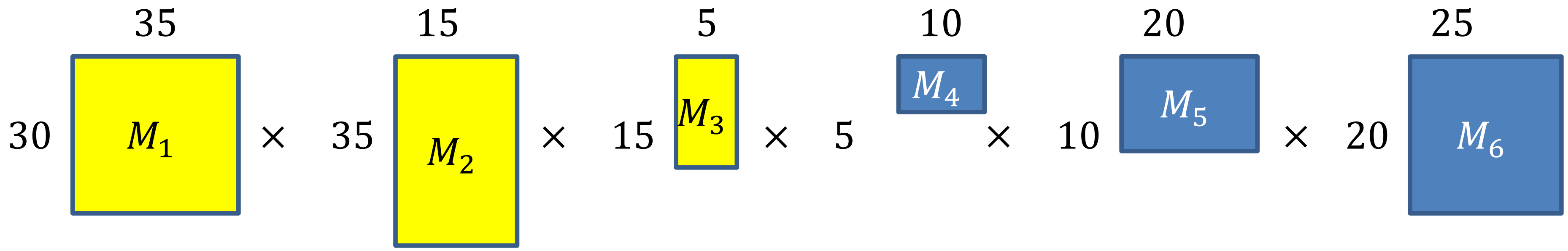


$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$i =$ \ $j =$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0	750		
4				0	1000	
5					0	5000
6						0

3. Select a good order for solving subproblems



$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

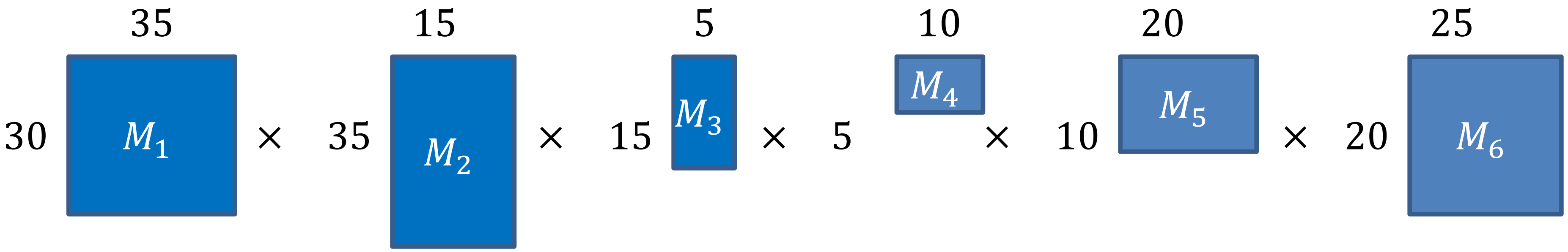
$$Best(i,i) = 0$$

$$r_1 r_2 c_3 = 30 \cdot 35 \cdot 5 = 5250$$
$$r_1 r_3 c_3 = 30 \cdot 15 \cdot 5 = 2250$$

$j =$ 1	2	3	4	5	6	$= i$
0	15750	7875				1
	0	2625				2
		0	750			3
			0	1000		4
				0	5000	5
					0	6

$$Best(1,3) = \min\begin{cases} \overset{0}{Best(1,1)} + \overset{2625}{Best(2,3)} + r_1 r_2 c_3 \\ \underset{15750}{Best(1,2)} + \underset{0}{Best(3,3)} + r_1 r_3 c_3 \end{cases}$$

3. Select a good order for solving subproblems



$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

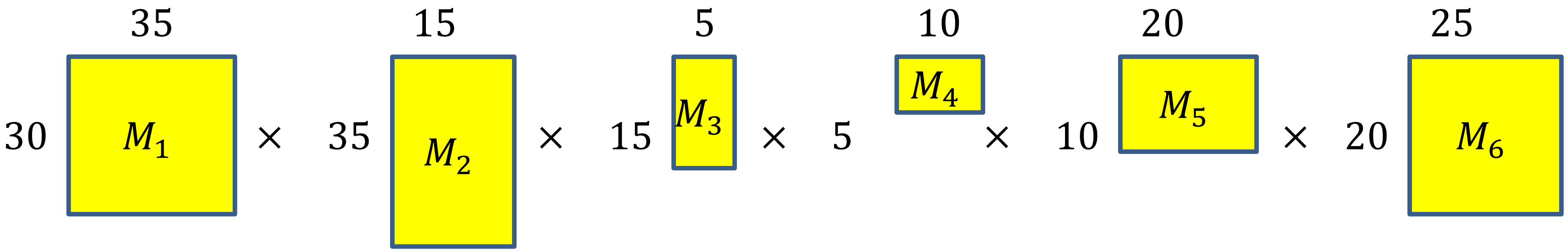
$$Best(i,i) = 0$$

$i =$ \ $j =$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625			
3			0	750		
4				0	1000	
5					0	5000
6						0

To find $Best(i,j)$: Need all preceding terms of row i and column j

Conclusion: solve in order of diagonal

Matrix Chaining



$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$i =$
0	15750	7875	9375	11875	15125	1
	0	2625	4375	7125	10500	2
		0	750	2500	5375	3
			0	1000	3500	4
				0	5000	5
					0	6

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ Best(1,3) + Best(4,6) + r_1 r_4 c_6 \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

Run Time

1. Initialize $Best[i,i]$ to be all 0s

$\Theta(n^2)$ cells in the Array

2. Starting at the main diagonal, working to the upper-right, fill in each cell using:
   1. $Best[i,i] = 0$
   2. $Best[i,j] = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$

$\Theta(n)$ options for each cell

Each "call" to Best() is a O(1) memory lookup

$\Theta(n^3)$ overall run time

Backtrack to find the best order

"remember" which choice of k was the minimum at each cell

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$= i$
0	15750	7875 1	9375	11875	15125 3	1
	0	2625	4375	7125	10500	2
		0	750	2500	5375	3
			0	1000	3500 5	4
				0	5000	5
					0	6

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ \boxed{Best(1,3) + Best(4,6) + r_1 r_4 c_6} \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

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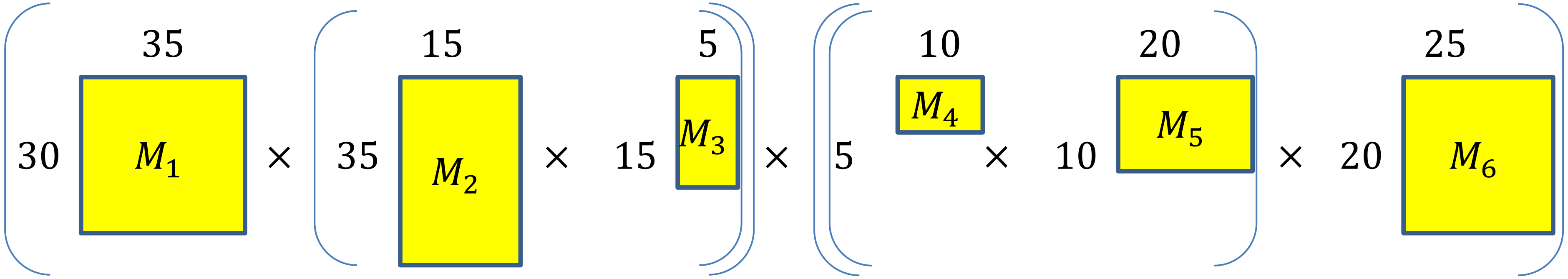
Matrix Chaining



$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$= i$
0	15750	7875 $_{1}$	9375	11875	15125 $_{3}$	1
	0	2625	4375	7125	10500	2
		0	750	2500	5375	3
			0	1000	3500 $_{5}$	4
				0	5000	5
					0	6

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ \boxed{Best(1,3) + Best(4,6) + r_1 r_4 c_6} \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

Storing and Recovering Optimal Solution

- Maintain table Choice[i,j] in addition to Best table
  - Choice[i,j] = k means the best "split" was right after M_k
  - Work backwards from value for whole problem, Choice[1,n]
  - Note: Choice[i,i+1] = i because there are just 2 matrices
- From our example:
  - Choice[1,6] = 3. So $[M_1\ M_2\ M_3]\ [M_4\ M_5\ M_6]$
  - We then need Choice[1,3] = 1. So $[(M_1)\ (M_2\ M_3)]$
  - Also need Choice[4,6] = 5. So $[(M_4\ M_5)\ M_6]$
  - Overall: $[(M_1)\ (M_2\ M_3)]\ [(M_4\ M_5)\ M_6]$

Dynamic Programming

- Requires Optimal Substructure
  – Solution to larger problem contains the solutions to smaller ones
- Idea:
  1. Identify the recursive structure of the problem
     - What is the "last thing" done?
  2. Save the solution to each subproblem in memory
  3. Select a good order for solving subproblems
     - "Top Down": Solve each recursively
     - "Bottom Up": Iteratively solve smallest to largest