Recurrence Relations

CS 4102: Algorithms

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Recurrence Relations

Solving Recurrence Relations

- Several (four) methods for solving:
  - Directly Solve
  - Substitution method
    - In short, guess the runtime and solve by induction
  - Recurrence trees
    - We won't see this in great detail, but a graphical view of the recurrence
    - Sometimes a picture is worth 2^{10} words!
  - "Master" theorem
    - Easy to find Order-Class for a number of common cases
    - Different variations are called different things, depending on the source

Directly Solving (or Iteration Method)

Directly Solve (unrolling the recurrence)

- For Mergesort:
  - $T(n) = 2*T(n/2) + n$
  - Do it on board →

Another Example!!

- Consider:
  - $T(n) = 3*T(n/4) + n$

Unroll the recurrence

- $T(n) = 3*T(n/4) + n$
- $T(n) = 3*[3*T(n/16)+n/4] + n$
- $\quad = 9T(n/16) + (7/4)n$

- $T(n) = 9T(n/16) + (7/4)n$
- $T(n) = 9[3T(n/64) + n/16] + (7/4)n$
- $T(n) = 27*T(n/64) + 9n/16 + 7n/4$
- $T(n) = 27*T(n/64) + 37n/16$ //Pattern??

- $T(n) = 3^d * T(n/4^d) + n * \sum(3/4)^{d-1}$ ← *sum from 1 to d*

Unroll the recurrence

- $T(n) = 3^d * T(n/4^d) + n * \sum(3/4)^{d-1}$

- We hit base case when:
  - $n/(4^d) = 1$
  - $n = 4^d$
  - $d = \log_4(n)$ //seem familiar??

Unroll the recurrence

- $T(n) = \mathbf{3^d * T(n/4^d)} + n * \sum(3/4)^d$

- Let's do one term at a time.
- $3^d * T(n/4^d)$
- $3^{\log4(n)} * T(1)$
- $3^{\log4(n)} = n^{\log4(3)}$ //huh? this is a log rule

Unroll the recurrence

- $T(n) = 3^d * T(n/(4^d)) + \mathbf{n * \sum(3/4)^{d-1}}$
- Let's do one term at a time.
  - $n * \sum(3/4)^{d-1}$ //note summation part approaches 4 as d grows
  - $n * \sum(3/4)^{d-1} <= 4*n = \Theta\mathbf{(n)}$

Unroll the recurrence

- $T(n) = 3^d * T(n/4^d) + n * \sum(3/4)^d$

- $T(n) = 3^{\log4(n)} + \Theta(n)$
- $T(n) = n^{\log4(3)} + \Theta(n)$ //log rules

- $T(n) = o(n) + \Theta(n)$

- ***$T(n) = \Theta(n)$***

Substitution Method

Iteration or Substitution Method

- Strategy
  - 1. Consider Mergesort Recurrence
    - T(n) = 2*T(n/2) + n
  - 2. Guess the solution
    - Let's go with n*log(n) **Remember logs are all base 2 (usually)
  - 3. Inductively Prove that recurrence is in proper order class
    - For n*log(n), we need to prove that T(n) <= c*n*log(n)
    - For some 'c' constant and for all n >= n0
    - Remember, we get to choose the 'c' and 'n0' values

  - Do it on board →

Substitution Method: Subtleties

- Consider:
  - T(n) = 2*T(n/2) + 1 T(1)=1
- Let's make our guess:
  - We are thinking O(n)
- Try to prove:
  - T(n) <= c*n

- What happens? How do we fix this issue?
- On board →

Substitution Method: Subtleties

- Consider:
  - $T(n) = 2*T(n/2) + 1$

Substitution Method: Subtleties

- Summary of the problem / issue:
  - T(n) = 2*T(n/2) + 1
  - T(n) <= 2(c*(n/2)) + 1
  - T(n) <= c*n + 1

- What is the issue here?
- c*n + 1 is TOO LARGE.
- Need to prove exact form of inductive hypothesis

Substitution Method: Subtleties

- Here is how we fix the issue. Subtract lower order term.
- Inductive Hypothesis:
  - T(n) <= c*n – d   //d is a constant term. Note c*n-d <= c*n

- Fix:
  - T(n) = 2*T(n/2) + 1
  - T(n) <= 2(c*(n/2) - d) + 1
  - T(n) <= c*n -2d + 1 <= c*n - d   //as long as d >= 1

Substitution Method: Another Pitfall

- Consider Mergesort recurrence again:
  - T(n) = 2*T(n/2) + n
- Let's make our guess:
  - We are thinking O(n) ← Note that this is INCORRECT!
- Try to prove:
  - T(n) <= c*n

- What happens?
- On board →

Substitution Method: Another Pitfall

- Consider Mergesort recurrence again:
  - $T(n) = 2*T(n/2) + n$

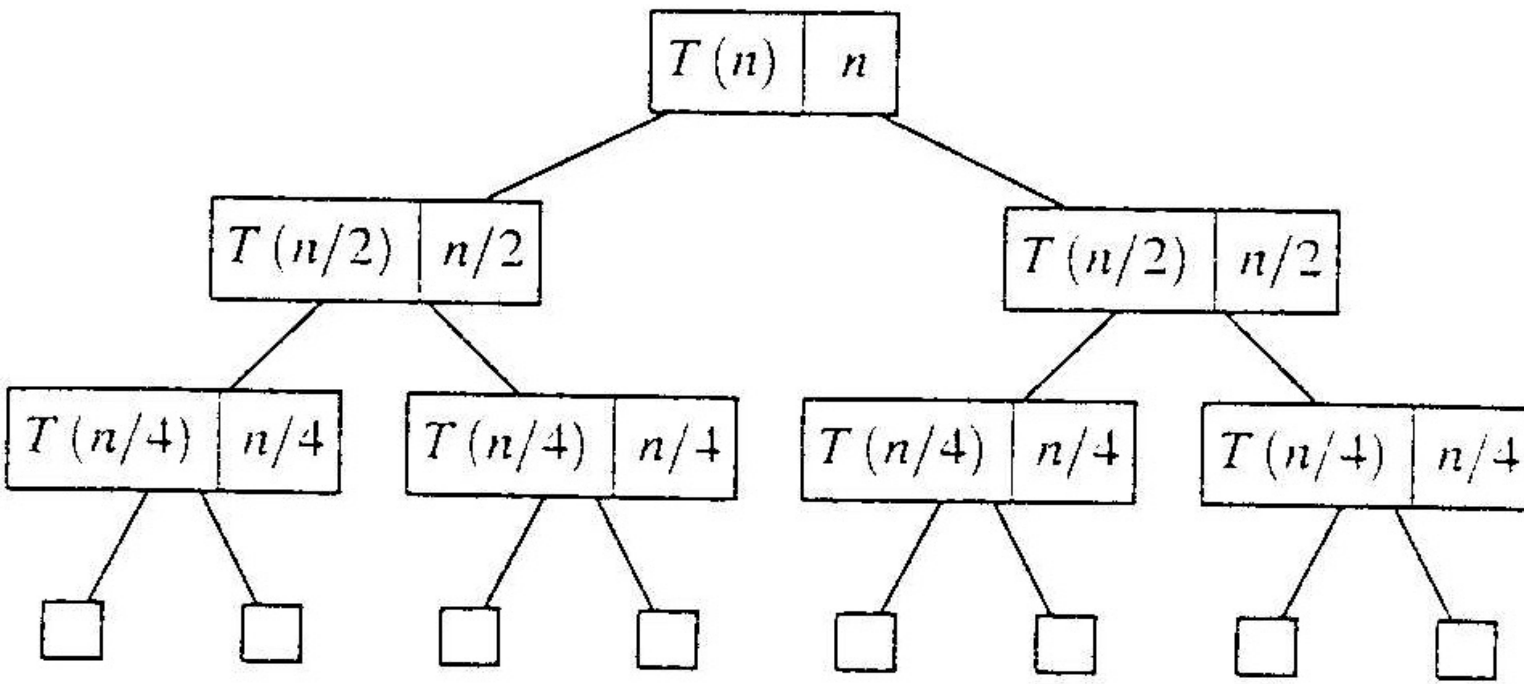
Substitution Method: Pitfall Example

- Attempt to prove:
  - T(n) = 2*T(n/2) + n
  - T(n) <= 2*(c*n/2) + n
  - T(n) <= c*n + n
- Again, need to prove EXACT form of inductive hypothesis.
- Subtracting off a lower order term won't help.
  - Why?

Recursion Tree Method

Recursion Tree Method

- Evaluate: T(n) = 2*T(n/2) + n
  - Work copy: T(k) = T(k/2) + T(k/2) + k
  - For k=n/2, T(n/2) = T(n/4) + T(n/4) + (n/2)
- [size| non-recursive cost]



Recursion Tree: Total Cost

- To evaluate the total cost of the recursion tree
  - sum all the non-recursive costs of all nodes
  - = Sum (rowSum(cost of all nodes at the same depth))
- Determine the maximum depth of the recursion tree:
  - For our example, at tree depth d the size parameter is $n/(2^d)$
  - the size parameter converging to base case, i.e. case 1
  - such that, $n/(2^d) = 1$,
  - $d = \lg(n)$
  - The rowSum for each row is n
- Therefore, the total cost, $T(n) = n \lg(n)$

The Master Theorem

The Master Theorem

- Given: a *divide and conquer* algorithm
  - An algorithm that divides the problem of size n into a subproblems, each of size n/b
  - Let the cost of each stage (i.e., the work to divide the problem + combine solved subproblems) be described by the function $f(n)$
- Then, the Master Theorem gives us a cookbook for the algorithm's running time
  - Some textbooks has a simpler version they call the "Main Recurrence Theorem"
  - We'll splits it into individual parts

The Master Theorem (from Cormen)

- If $T(n) = a\,T(n/b) + f(n)$
  - then let $k = \lg a / \lg b = \log_b(a)$ (critical exponent)

- Then three common cases:
  - If $f(n) \in O(n^{k-\varepsilon})$ for some positive ε, then $T(n) \in \Theta(n^k)$
  - If $f(n) \in \Theta(n^k)$ then $T(n) \in \Theta(\, f(n) \log(n)\,) = \Theta(n^k \log(n))$
  - If $f(n) \in \Omega(n^{k+\varepsilon})$ for some positive ε, and $a\, f(n/b) \leq c\, f(n)$ for some $c < 1$ and sufficiently large n, then $T(n) \in \Theta(f(n))$

- Note: none of these cases may apply

Using the Master Theorem

- $T(n) = 9T(n/3) + n$
  - $A = 9$, $b = 3$, $f(n) = n$

- Master Theorem
  - $k = \lg 9 / \lg 3 = \log_3 9 = 2$
  - Since $f(n) = O(n^{\log_3 9 - \varepsilon})$, where $\varepsilon=1$, case 1 applies:

$$T(n) \in \Theta(n^k)$$

  - Thus the solution is $T(n) = \Theta(n^2)$ since $k=2$

Problems to Try

- Can you use a theorem on these?
- Assume $T(1) = 1$

- $T(n) = T(n/2) + \lg n$
- $T(n) = T(n/2) + n$
- $T(n) = 2T(n/2) + n$ (like Mergesort)
- $T(n) = 2T(n/2) + n \lg n$

More Master Theorem Examples

Problems to Try

- Let's try these?

- $T(n) = 7T(n/3) + n^2$
- $T(n) = 3T(n/3) + n/2$
- $T(n) = 4T(n/2) + n / \log(n)$
- $T(n) = 3T(n/3) + n / \log(n)$

Problems to Try: Solutions

- $T(n) = 7T\left(\frac{n}{3}\right) + n^2$
  - $k = \log_3(7) = 1.77$
  - $n^k = n^{1.77}$
  - Case 3: n^2

$f(n) = n^2$

regularity: $7 * f\left(\frac{n}{3}\right) \leq c * n^2$

$7 * \frac{n^2}{9} \leq c * n^2$

$\frac{7}{9}n^2 \leq c * n^2$

//YES

Problems to Try: Solutions

- $T(n) = 3T\left(\frac{n}{3}\right) + \frac{n}{2}$
  - $k = \log_3(3) = 1$
  - $n^k = n$ $f(n) = \frac{n}{2}$
  - Case 2: nlogn

Problems to Try: Solutions

- $T(n) = 4T\left(\frac{n}{2}\right) + \frac{n}{\log(n)}$
- $k = \log_2(4) = 2$
  - $n^k = n^2$ $f(n) = \frac{n}{\log(n)}$
  - Case 1: n^2

Problems to Try: Solutions

- $T(n) = 3T\left(\frac{n}{3}\right) + \frac{n}{\log(n)}$
  - $k = \log_3(3) = 1$
  - $n^k = n$ $\quad f(n) = \frac{n}{\log(n)}$
  - Case 1 doesn't apply because f(n) not polynomially smaller
  - e.g., n / log(n) !<= n^0.99 for large n

Solutions

Solutions to problems that aren't directly in the slides above

Directly Solve (unrolling the recurrence)

- For Mergesort:
  - $T(n) = 2*T(n/2) + n$
  - Do it on board →

Directly Solve (unrolling the recurrence)

T(n) = 2*T(n/2) + n

T(n) = 2*[2 * T(n/4) + n/2] + n //unroll one level
= 4*T(n/4) + 2n

= 4*[2*T(n/8) + n/4] + 2n //unroll another level
= 8*T(n/8) + 3n

= 8*[2*T(n/16) + n/8] + 3n //one more time
= 16*T(n/16) + 4n

Directly Solve (unrolling the recurrence)

T(n) = 4*T(n/4) + 2n

= 8*T(n/8) + 3n

= 16*T(n/16) + 4n

= … //what is the general pattern??

= $2^d * T\left(\frac{n}{2^d}\right) + dn$ //where d is depth of recursion

Directly Solve (unrolling the recurrence)

T(n) = $2^d * T\left(\frac{n}{2^d}\right) + dn$ //where d is depth of recursion

$\frac{n}{2^d} = 1$ //when do we hit T(1)

$n = 2^d$

$d = \log_2 n$ //recursion ends when d is log(n)

T(n) = $2^d * T\left(\frac{n}{2^d}\right) + dn$ //sub back in for d

T(n) = $2^{\log_2(n)} * T\left(\frac{n}{2^{\log_2(n)}}\right) + \log_2(n) * n$

T(n) = $n * T(1) + \log_2(n) * n$

***T(n)* = $\boldsymbol{n} + \boldsymbol{log_2(n)} * \boldsymbol{n} = \boldsymbol{\Theta}(\boldsymbol{nlog(n)})$**

Iteration or Substitution Method

- T(n) = 2*T(n/2) + n
  - Guess n*log(n)

- $T(n) \leq \ c * n * \log_2(n)$

- Base case (n=2):
  - $T(2) \leq c * 2 * \log_2(2)$
  - 2*T(1)+2 $\leq c * 2 * 1$
  - 4 $\leq 2c$ //true if c >= 2

Iteration or Substitution Method

- T(n) = 2*T(n/2) + n
  - Guess n*log(n)
- $T(n) \leq \ c * n * \log_2(n)$

- Inductive Hypothesis:
  - Assume for all $k < n$ that $T(k) \leq c * k * \log_2(k)$

Iteration or Substitution Method

- T(n) = 2*T(n/2) + n
  - Guess n*log(n)

- Inductive Step:
  - $T(n) = 2 * T\left(\frac{n}{2}\right) + n$
  - $T(n) \leq 2 * (c * \frac{n}{2} * \log_2(\frac{n}{2})) + n$
  - $T(n) \leq cn * (\log_2(n) - \log_2(2)) + n$
  - $T(n) \leq cn * (\log_2(n) - 1) + n$
  - $T(n) \leq c * n * log_2(n) - cn + n \leq \mathbf{c * n * log_2(n)}$

//if c >= 1

Problems to Try

- Can you use a theorem on these?
- Assume $T(1) = 1$

- $T(n) = T(n/2) + \lg n$
- $T(n) = T(n/2) + n$
- $T(n) = 2T(n/2) + n$ (like Mergesort)
- $T(n) = 2T(n/2) + n \lg n$

Problems to Try

- $T(n) = T\left(\frac{n}{2}\right) + \lg(n)$
  - $k = \log_2(1) = 0$
  - $n^0 = 1$ $\qquad f(n) = \lg(n)$
- Case 3 does not apply!

- $T(n) = T\left(\frac{n}{2}\right) + n$
  - $k = \log_2(1) = 0$
  - $n^0 = 1$ $\qquad f(n) = n$

  - Case 3: $T(n) = \Theta(n)$ $\qquad 1 * \left(\frac{n}{2}\right) \leq c * n$ //YES

Problems to Try

- $T(n) = 2T\left(\frac{n}{2}\right) + n$ (like Mergesort)
  - $k = \log_2(2) = 1$
  - n^1 $f(n) = n$
  - Case 2: $T(n) = \Theta(nlog(n))$
- $T(n) = 2T\left(\frac{n}{2}\right) + nlog(n)$
  - k = 1
  - n^1 $f(n) = nlog(n)$
  - $nlog(n) \geq c * n^{1+\epsilon}$ //NO! not polynomially smaller!
  - Master theorem cannot be used