CS4102 Algorithms

Fall 2021 – Horton and Floryan

These are Horton's version of the slides, used in his lecture.

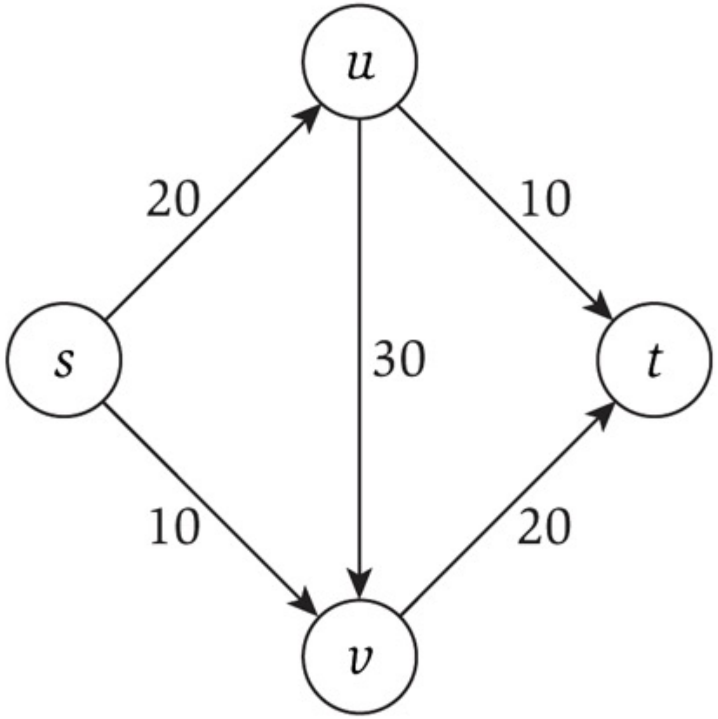
Network Flow, Ford-Fulkerson

In your textbook

- CLRS 26.1 and 26.2
- Includes simple solutions to the following “complications”
  - What if (u,v) and (v,u) are in the flow graph?
    - Called “Antiparallel” edges – easy to adjust for this, example later
  - What if we need >1 source? >1 sink?

Flow networks

- Consider a flow network, which is a specialized directed graph with:
  - A single source node s
  - A single terminus node t
  - Capacities on each edge
    - That must be integer!
- What is the maximum flow you can send from s to t?

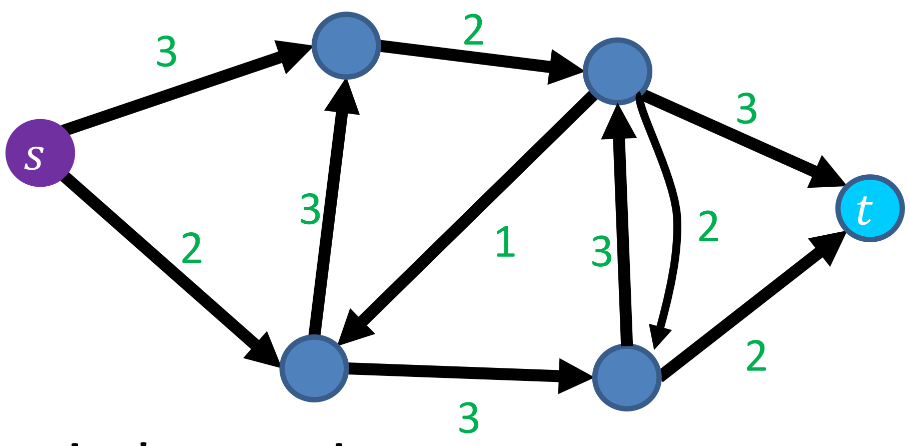


Applications

- Transportation networks
  - How many people can be routed?
- Computer networks
- Electrical distribution
- Water distribution

- Note that all these applications have multiple sources and multiple sinks!
  - Whereas the flow networks we study do not, yet

Flow Network

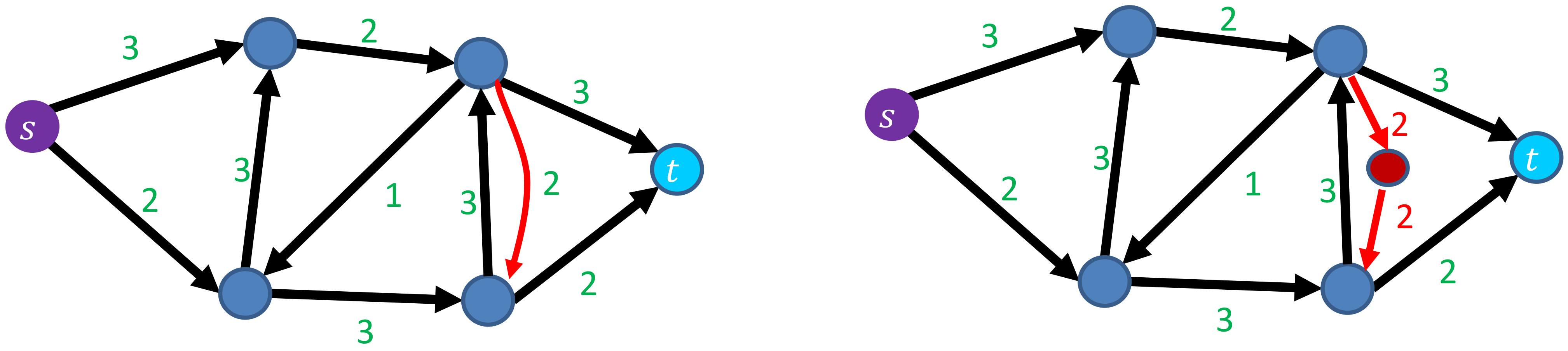


Graph $G = (V, E)$
Source node $s \in V$
Sink node $t \in V$
Edge Capacities $c(e) \in$ Positive whole* numbers
If $(u, v) \in E$ then $(v, u) \notin E$ (Note our example here violates this!)

Max flow intuition: If s is a faucet, t is a drain, and s connects to t through a network of pipes with given capacities, what is the maximum amount of water which can flow from the faucet to the drain?

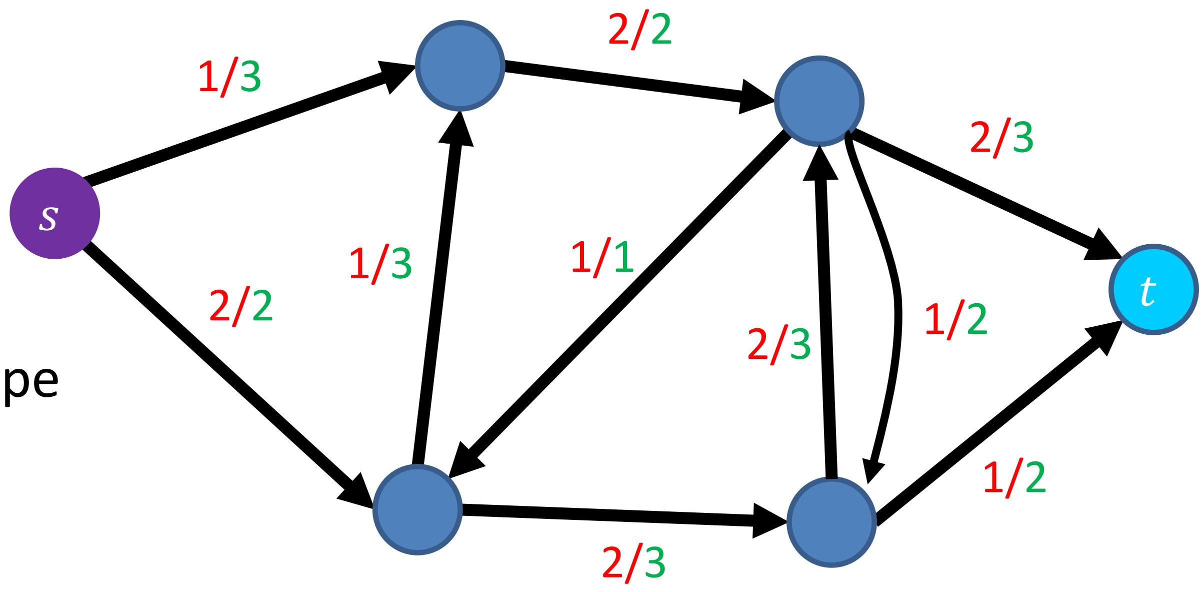
Flow Network: Antiparallel Edges

Easy adjustment to remove antiparallel edges and have equivalent flow graph: add intermediate node



(Note: our later examples use graph on the left without this adjustment.)

Flow



Flow/Capacity

- Assignment of values to edges
  - $f(e) = n$
  - E.g. n units of water going through that pipe
- Capacity constraint
  - $f(e) \leq c(e)$
  - Flow cannot exceed capacity
- Flow constraint
  - $\forall v \in V - \{s, t\}, inflow(v) = outflow(v)$
  - $inflow(v) = \sum_{x \in V} f(v, x)$
  - $outflow(v) = \sum_{x \in V} f(x, v)$
  - Water going in must match water coming out
- Flow of G: $|f| = outflow(s) - inflow(s)$
  - Net outflow of s

3 in example above

Let's Make Some Rules

- Source node has NO incoming flow
- Terminal (sink) node has NO outgoing flow

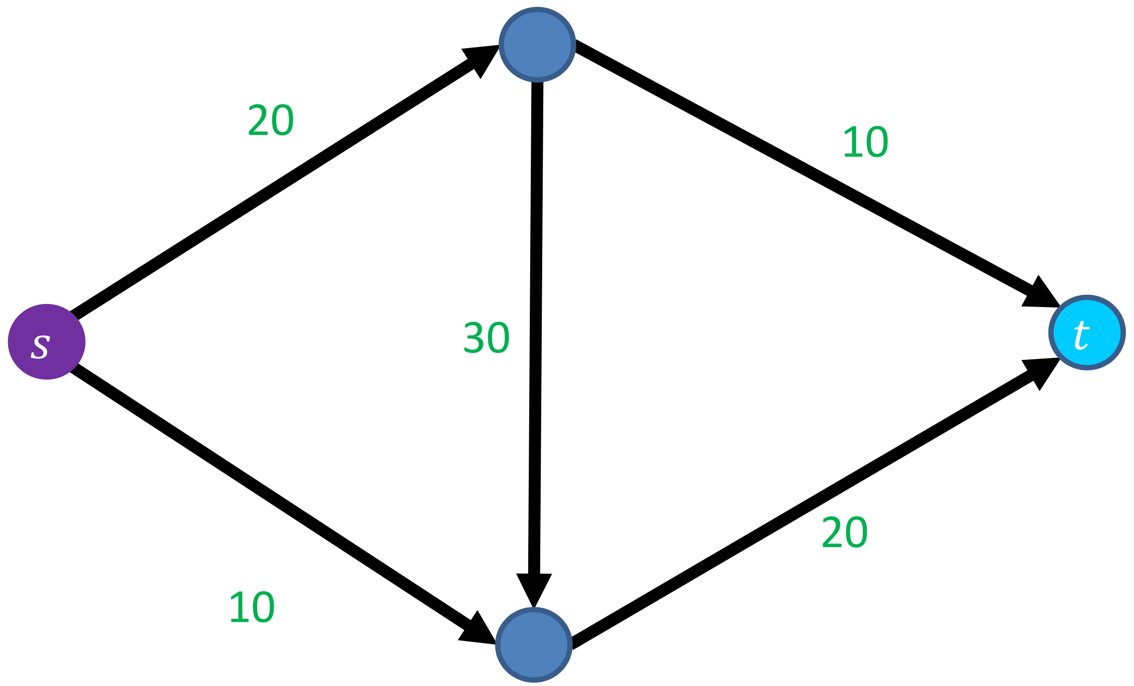
- Internal nodes has net zero flow
  - all units of flow going in must be going out as well

- No edge is over capacity

- GOAL: Find the maximum flow that can be "pushed" through the network
  - I.e. maximize: $|f| = outflow(s) - inflow(s)$

How to Solve This? This Greedy doesn't work

Saturate Highest Capacity Path First



Greedy doesn’t work

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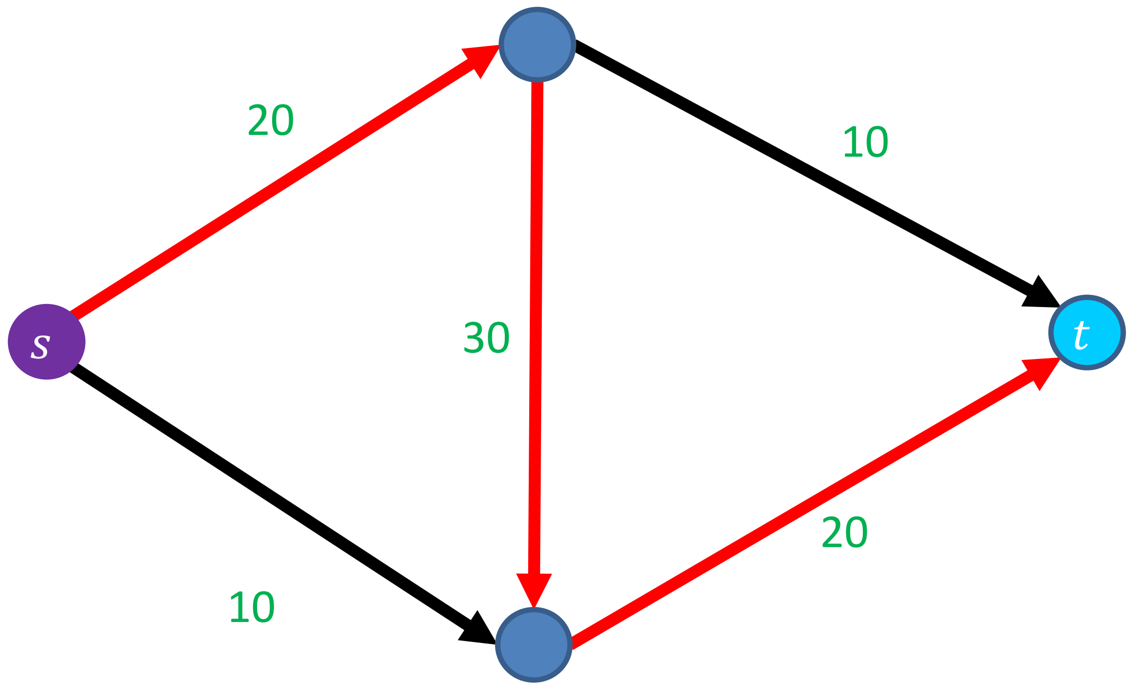

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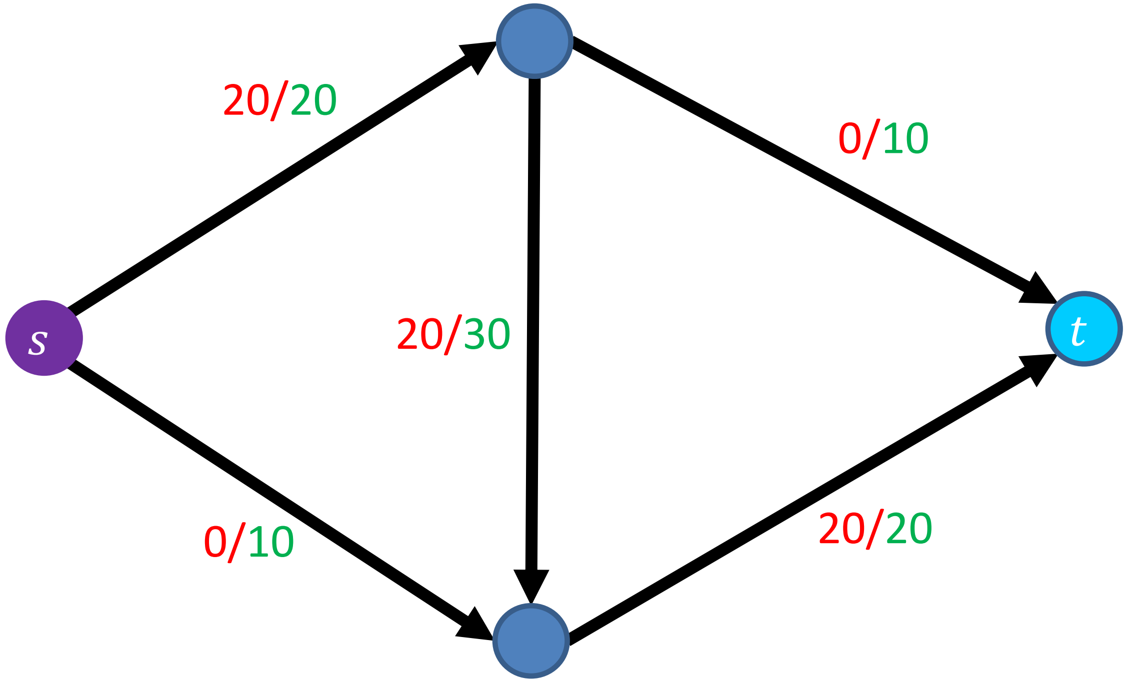
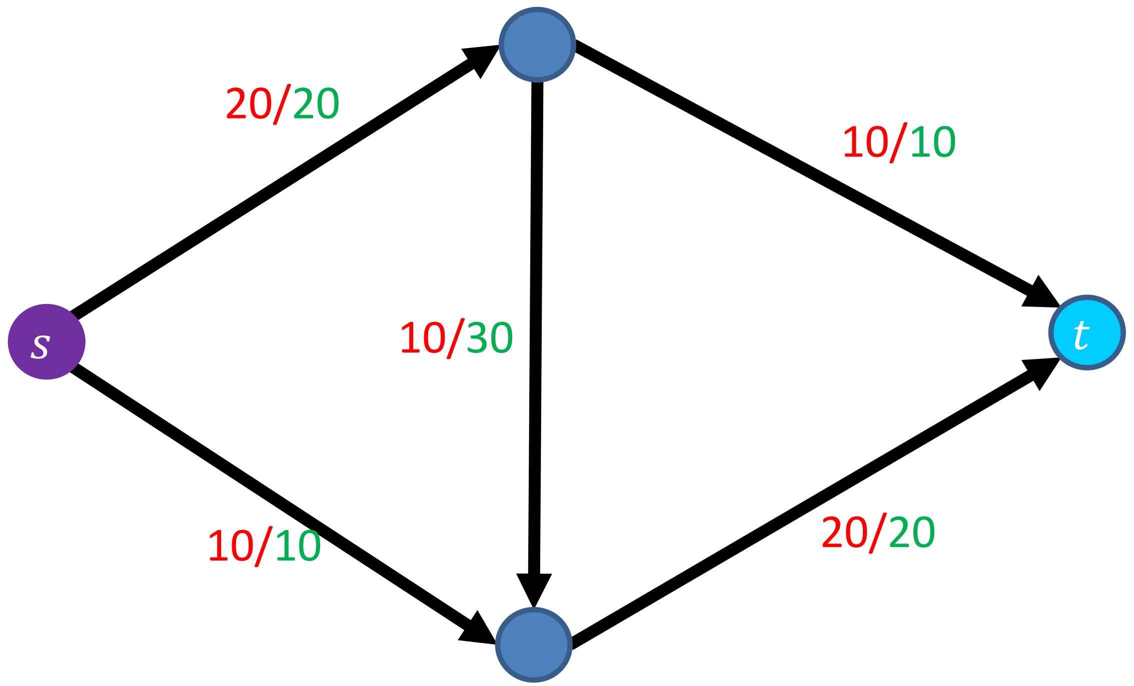


Overall Flow: $|f| = 20$

Greedy doesn't work

Better Solution



Overall Flow: $|f| = 30$

Ford-Fulkerson: Algorithm overview

- Iterative algorithm: push some flow along some path at each step
- Model or record the *residual* capacities
  - how much capacity is left after taking into account how much flow is going through that edge at this time
- Find a path from *s* to *t* such that the minimum residual capacity of an edge on that path is greater than zero
  - Since each value is an integer, it must be 1 or more
- Update the residual capacities after taking into account this new flow
- Repeat until no more such paths are found

Algorithm notation

- $f(u,v)$: the flow on the edge from u to v
- $f(v,u)$: the backflow on the edge from v to u
- $c(u,v)$: the capacity on the edge from u to v
- $c_f(u,v)$: the *residual* capacity on the edge from u to v
- G_f is the graph where the edges weights are the residual capacities
  - THIS is usually the graph we actually use when running the algorithm we are about to see.

Backflow

- Each edge has forward flow and backflow
  - The two must always be “inverses” of each other!
  - I.e. they sum to the total capacity for that edge
- This allows for modeling of flow “returning” along a given edge

- One way to think about this:
  - How much of the forward flow we could “un-do”

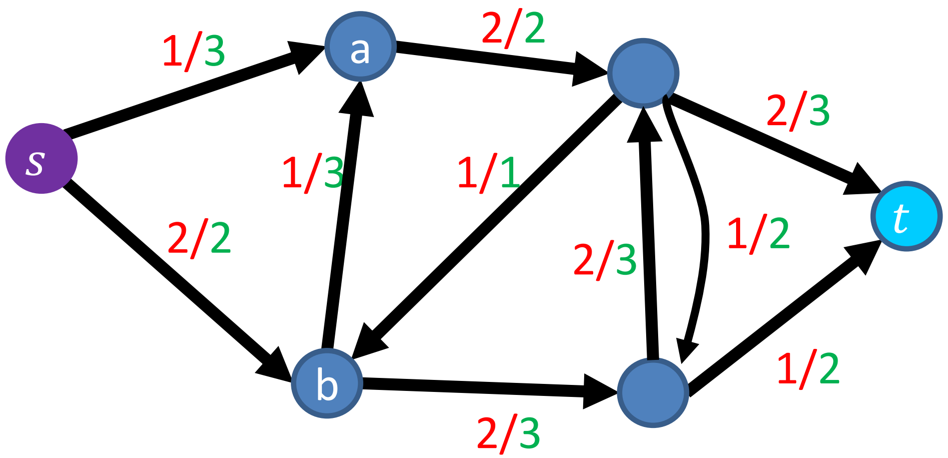
Residual Graph G_f

- Keep track of net available flow along each edge
- "Forward edges": weight is equal to available flow along that edge in the flow graph
  - $w(e) = c(e) - f(e)$
- "Back edges": weight is equal to backflow along that edge in the flow graph
  - $w(e) = f(e)$

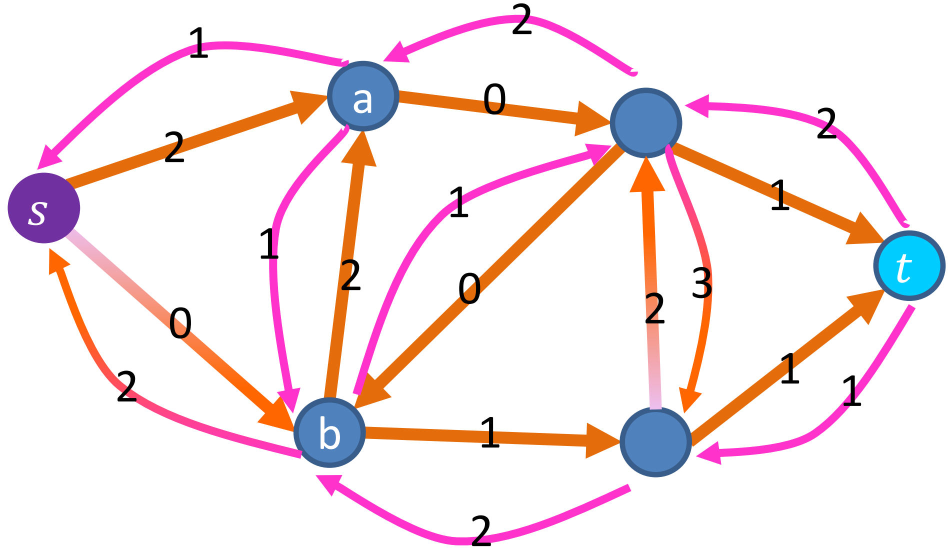
Flow I *could* add

Flow I *could* remove

Flow Graph G

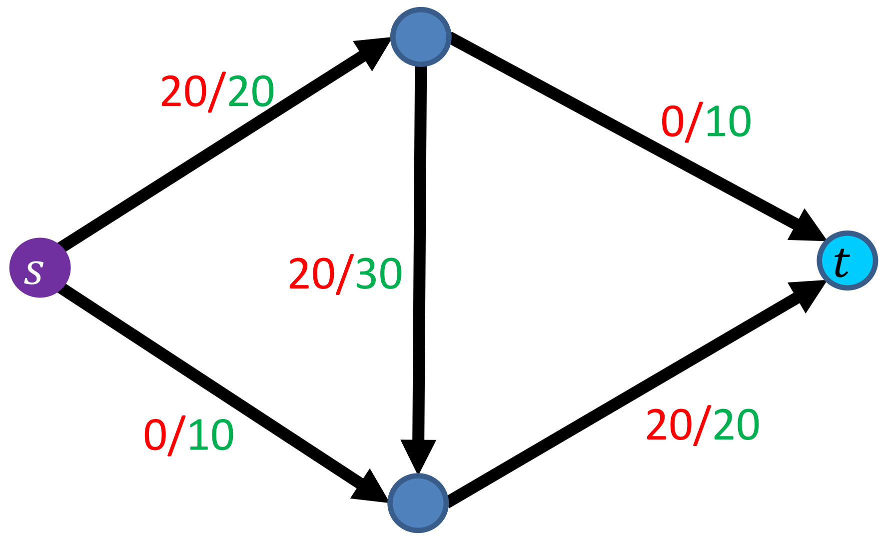


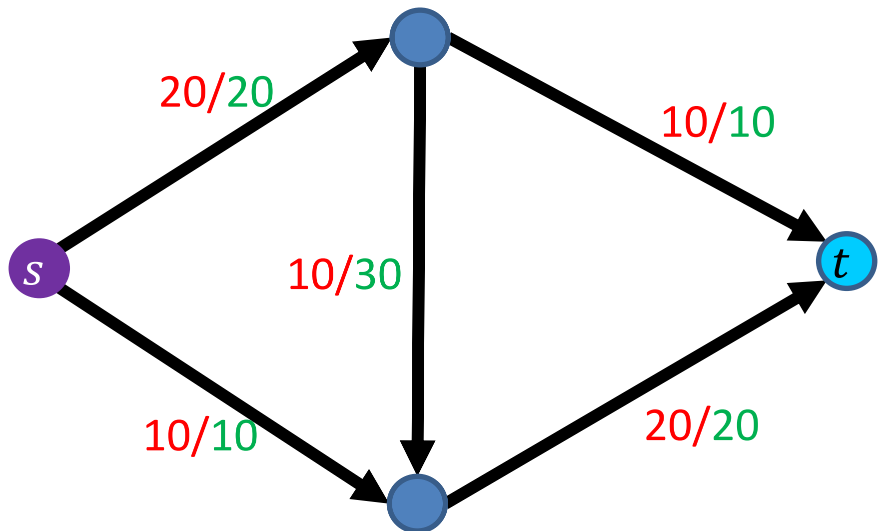
Residual Graph G_f



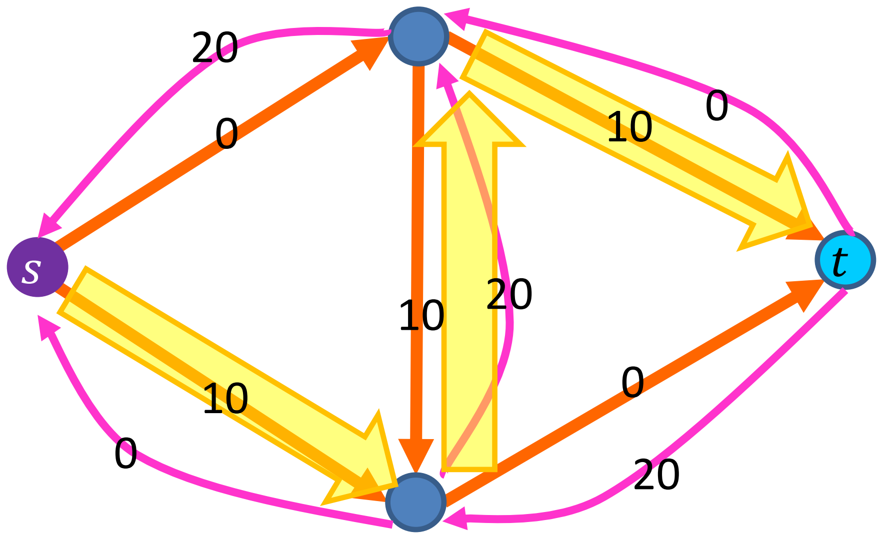
Residual Graphs Example

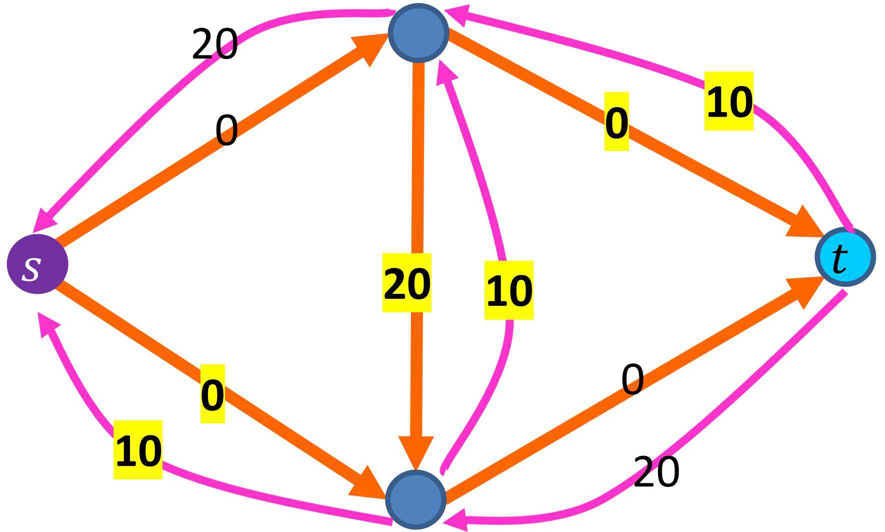
Flow Graph





Residual Graph





Ford-Fulkerson Algorithm

Define an **augmenting path** to be a path from $s \to t$ in the residual graph G_f (using edges of non-zero weight)

Overview: Repeatedly add the flow of any augmenting path

Ford-Fulkerson max-flow algorithm:

- Initialize $f(e) = 0$ for all $e \in E$
- Construct the residual network G_f
- While there is an augmenting path p in G_f:
  - Let $c = \min_{u,v \in p} c_f(u, v)$
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Ford-Fulkerson approach: take <u>any</u> augmenting path (will revisit this later)

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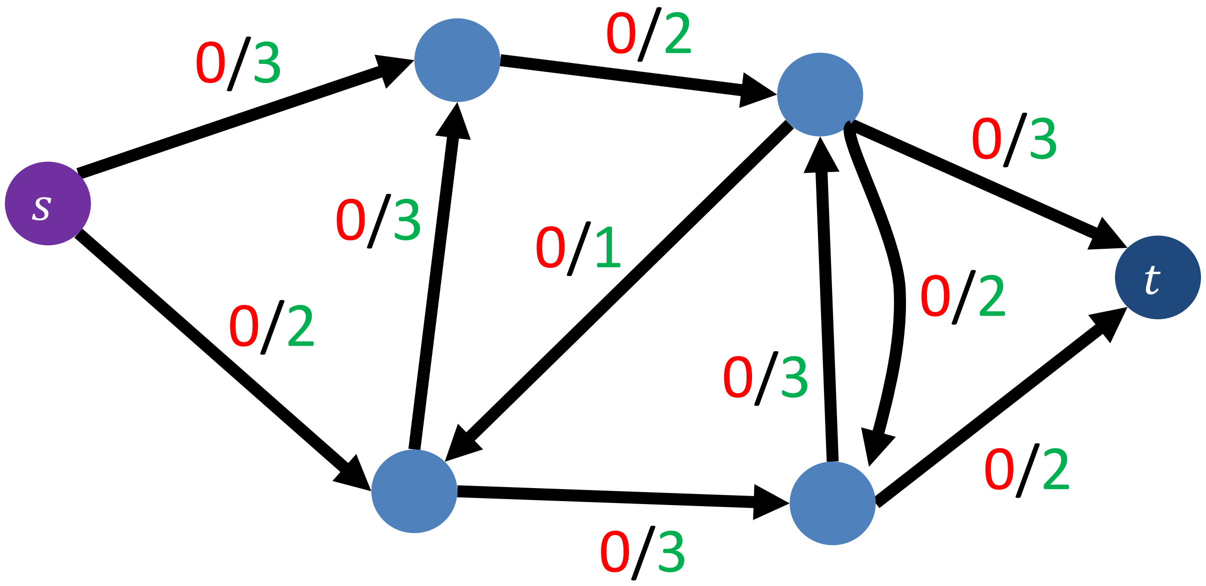
Ford-Fulkerson approach: take <u>any</u> augmenting path (will revisit this later)

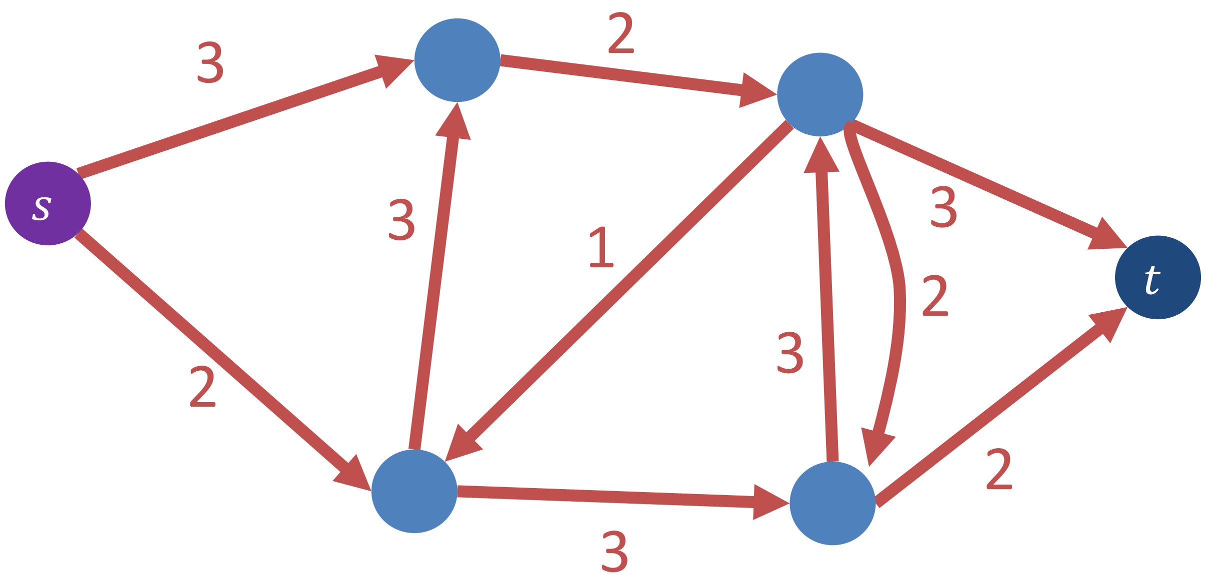
($c_f(u, v)$ is the weight of edge (u, v) in the residual network G_f)

Ford-Fulkerson Algorithm: Updating G_f

1. $f(u,v) = 0$ for all edges (u,v)
2. While there is an "augmenting" path p from s to t in G_f such that $c_f(u,v) > 0$ for all edges $(u,v) \in p$
   a. Find $c_f(p) = \min\{c_f(u,v) \mid (u,v) \in p\}$
   b. For each edge $(u,v) \in p$
      i. $f(u,v) = f(u,v) + c_f(p)$ send flow along the path
      ii. $f(v,u) = f(v,u) - c_f(p)$ send backflow the other way

Ford-Fulkerson Example

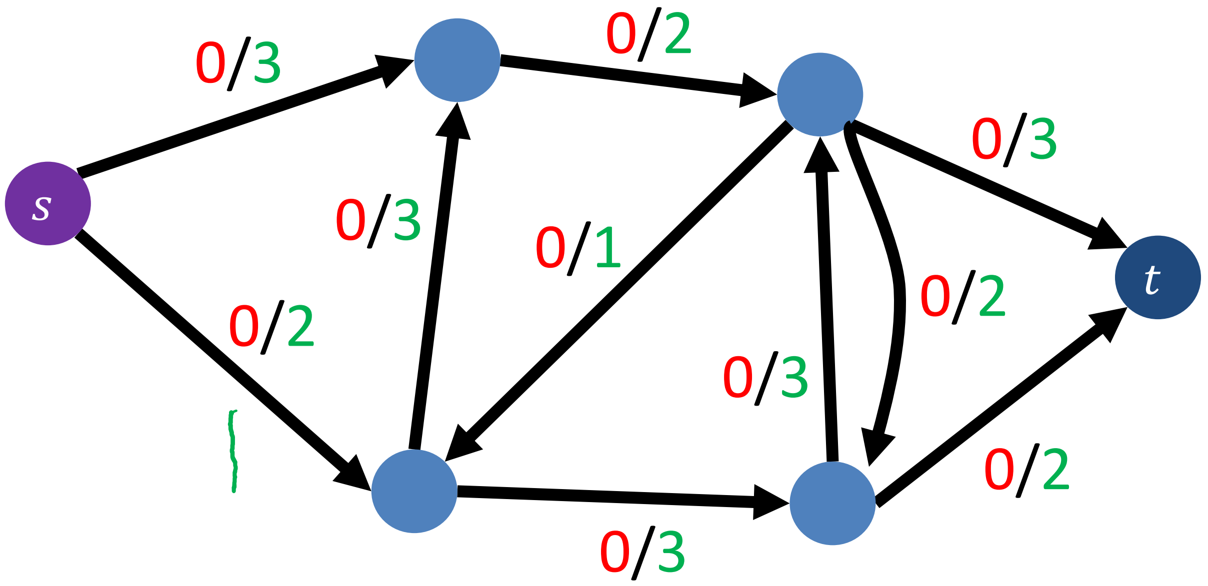




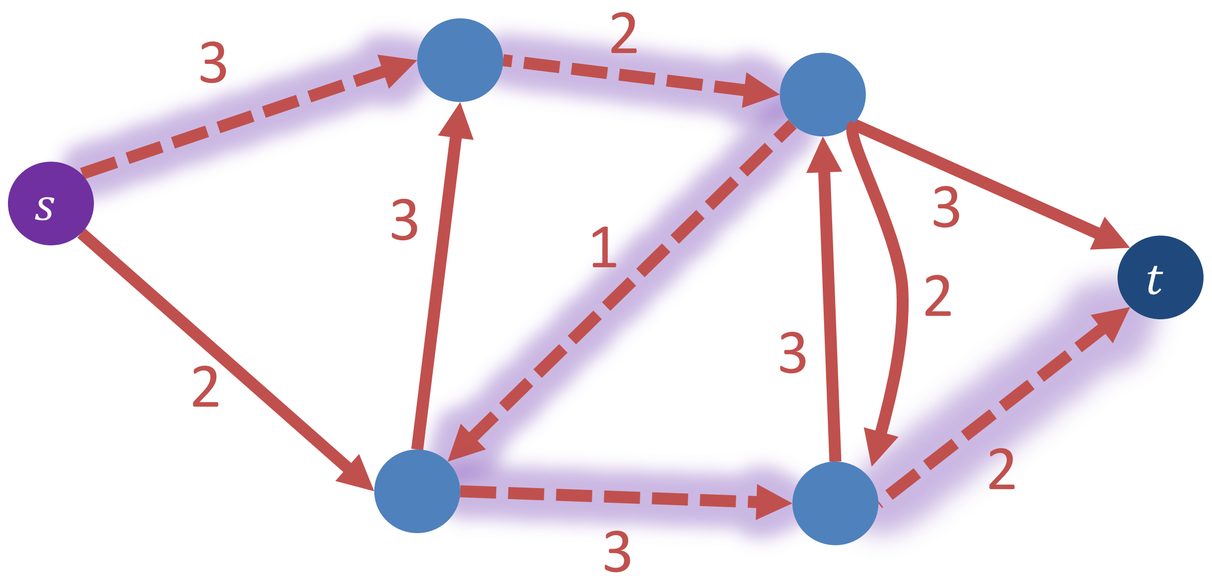
Initially: $f(e) = 0$ for all $e \in E$

Residual graph G_f

Ford-Fulkerson Example

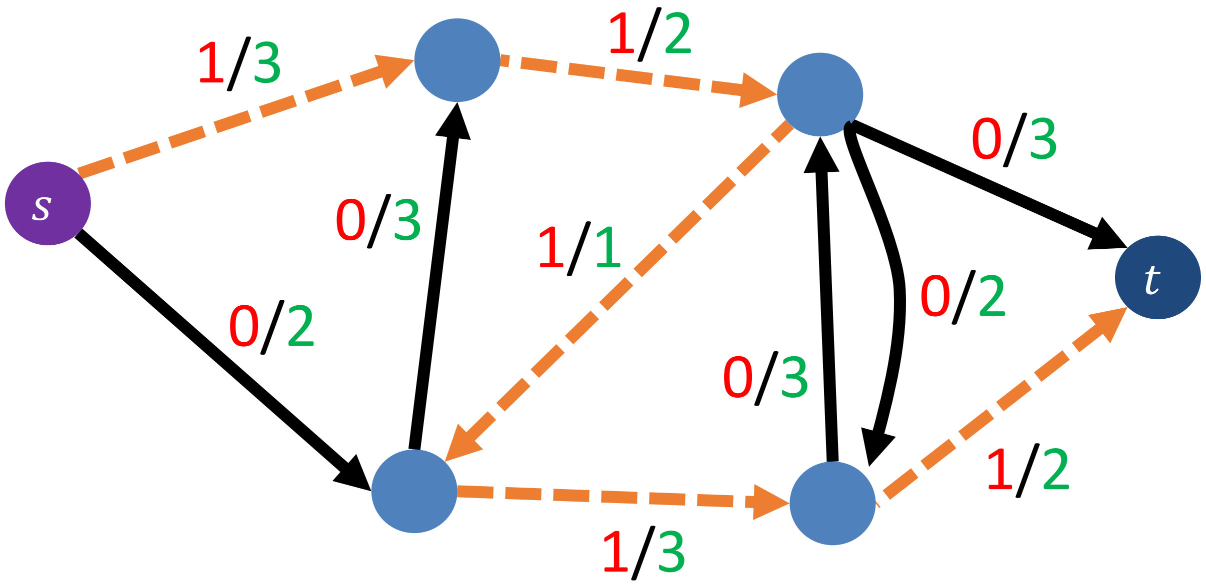


Increase flow by 1 unit



Residual graph G_f

Ford-Fulkerson Example



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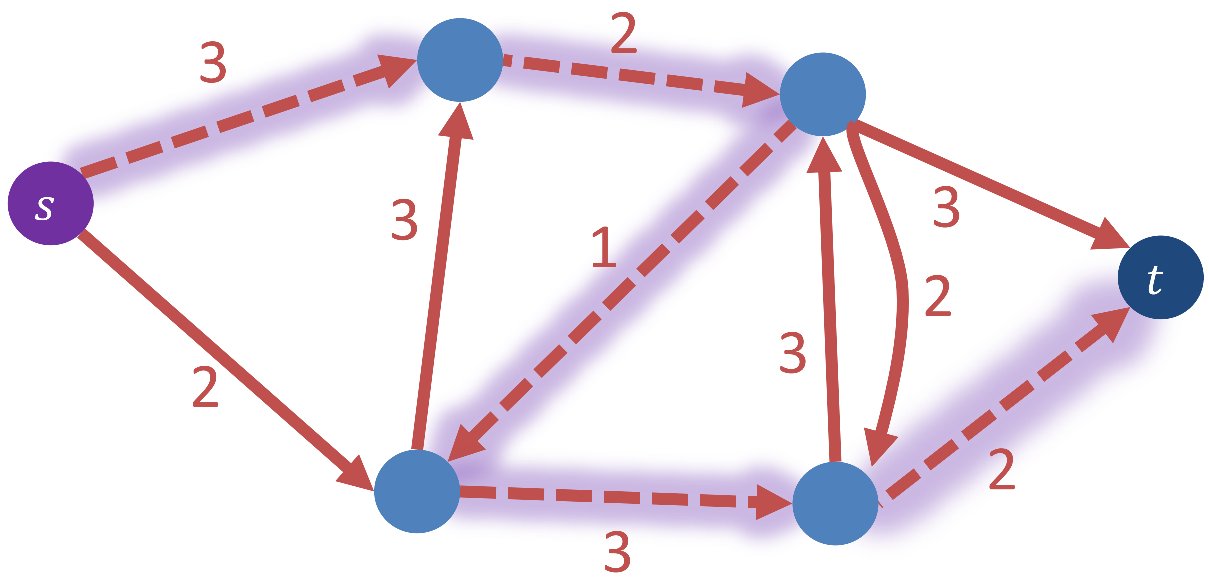

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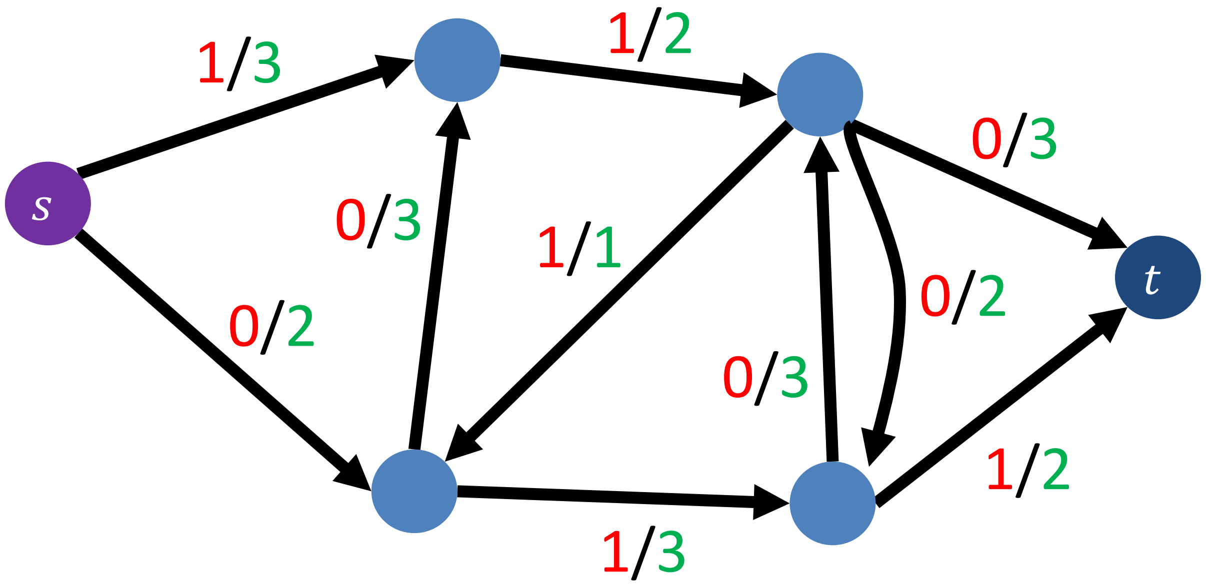

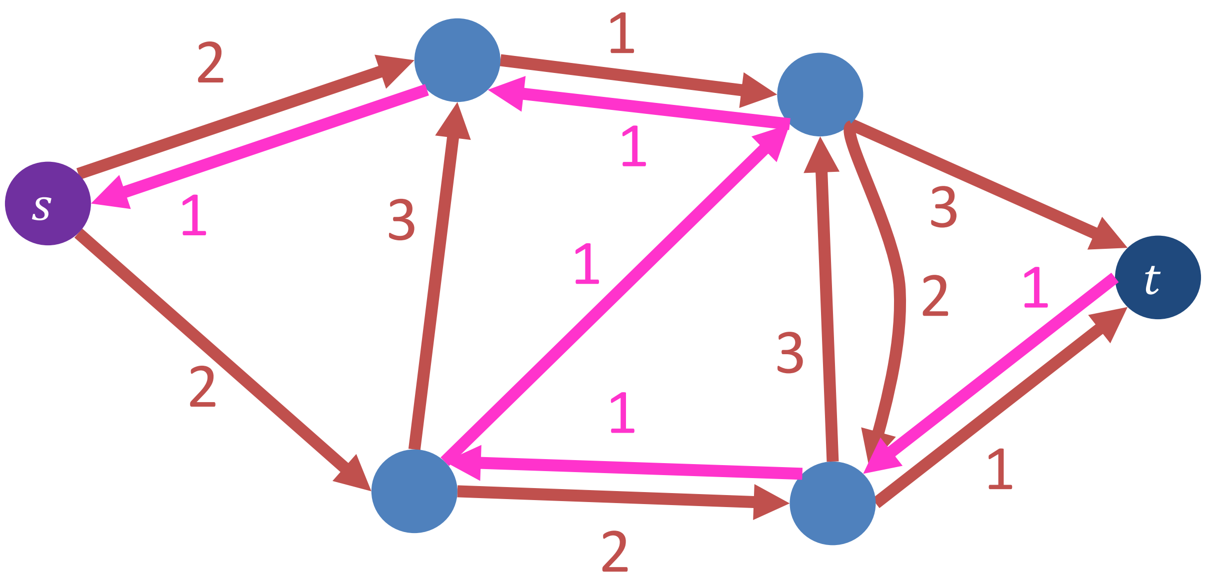

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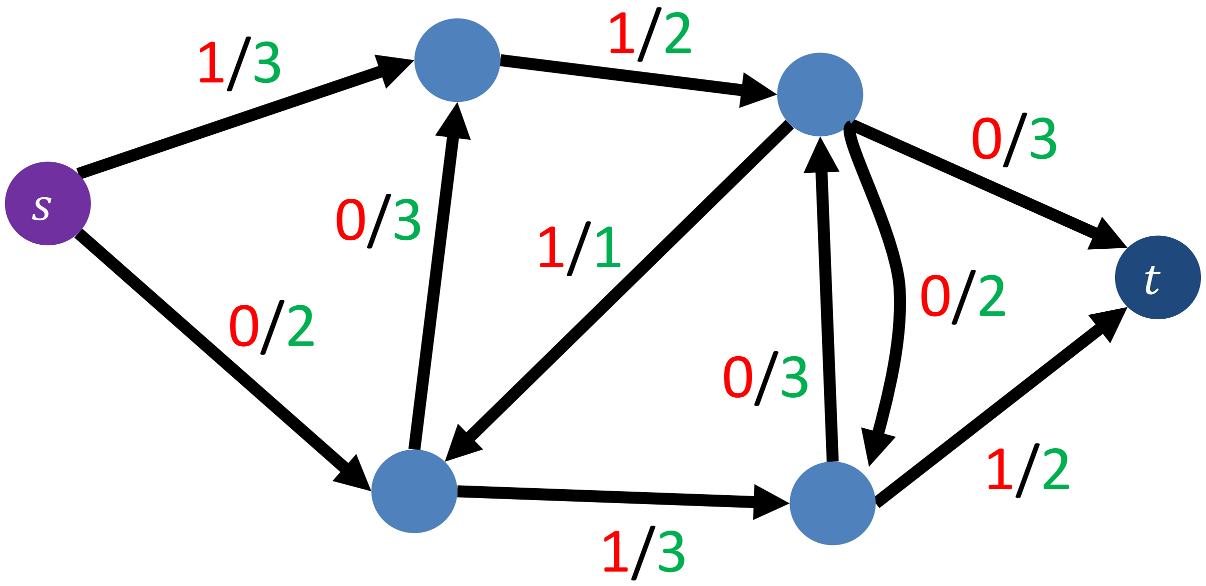

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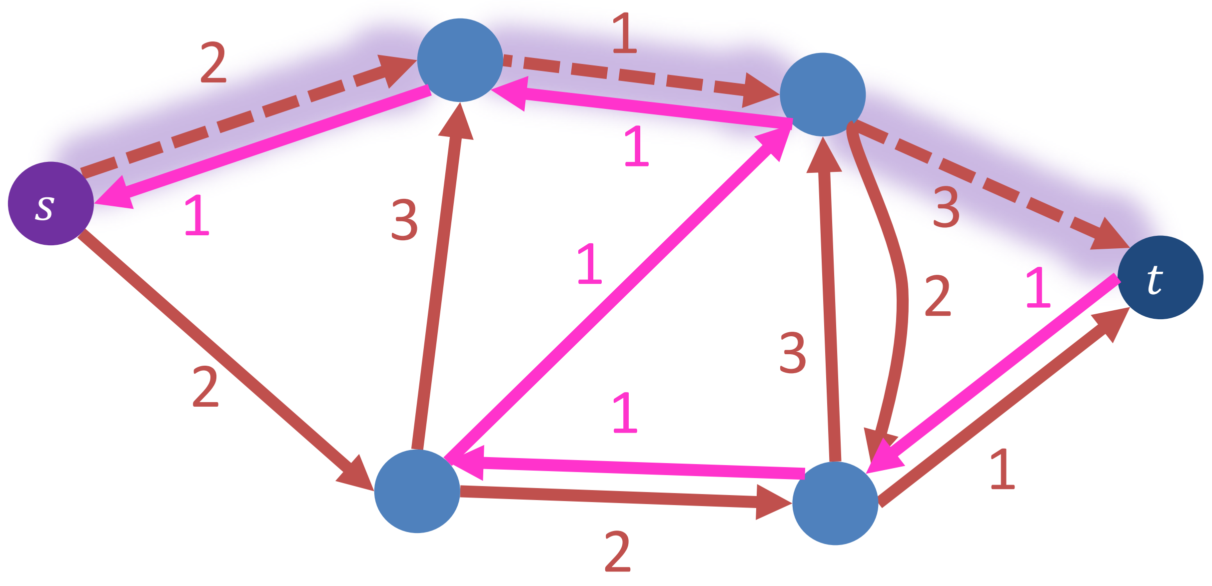

Residual graph G_f

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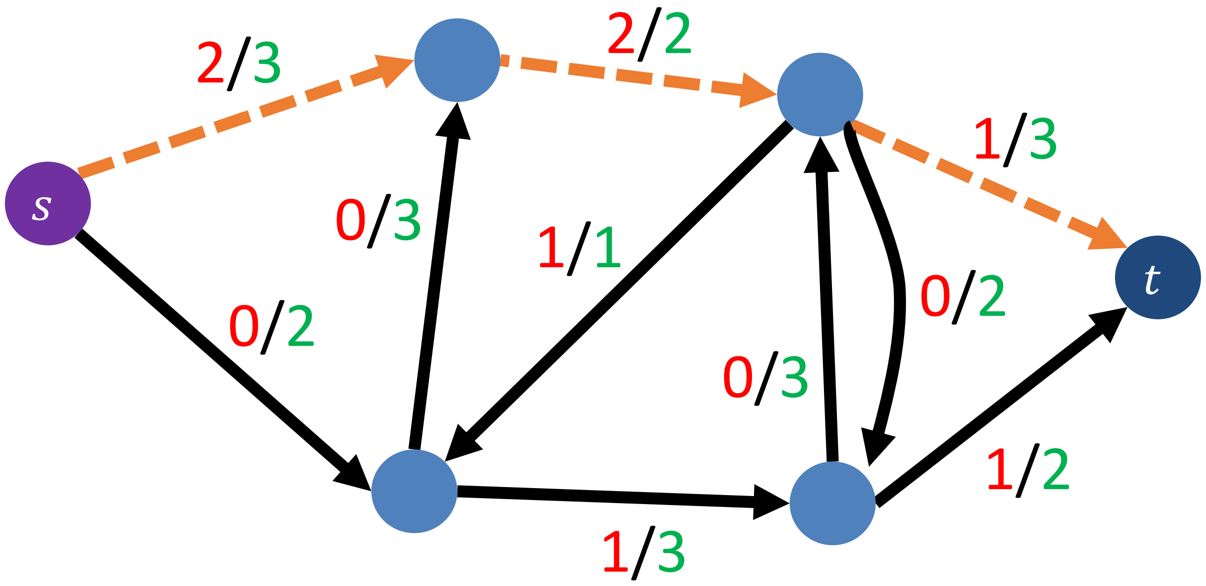

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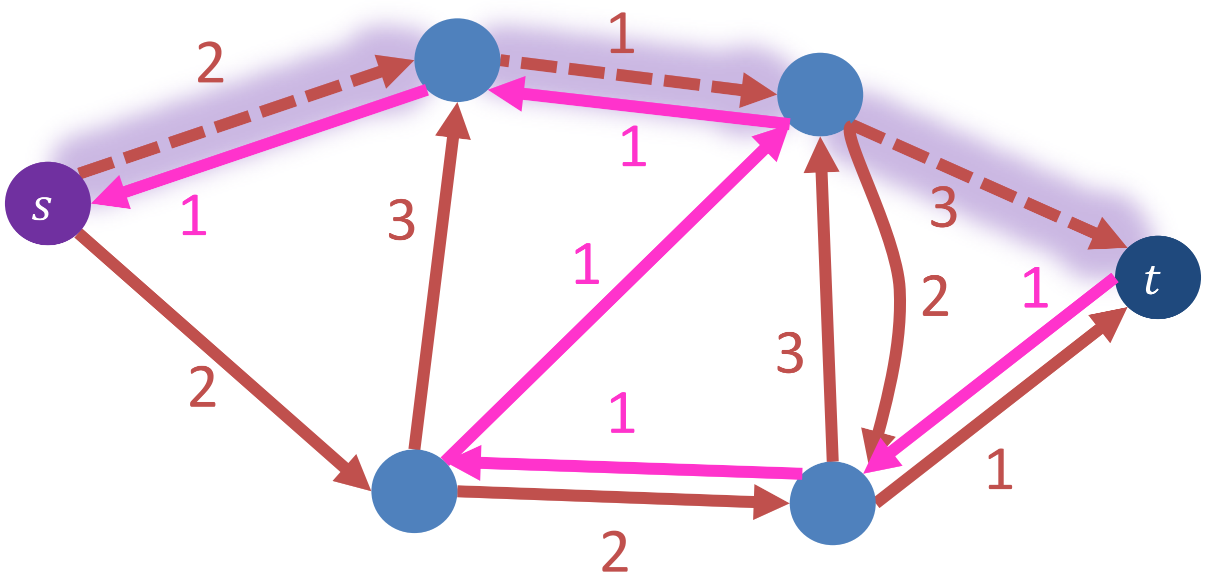

Residual graph G_f

Ford-Fulkerson Example

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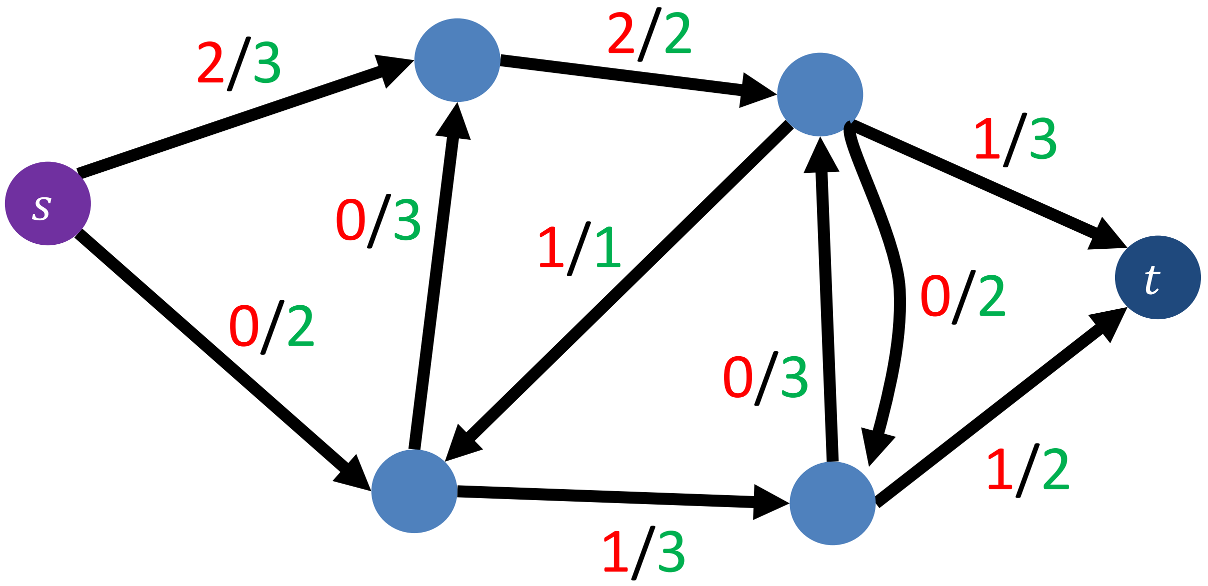

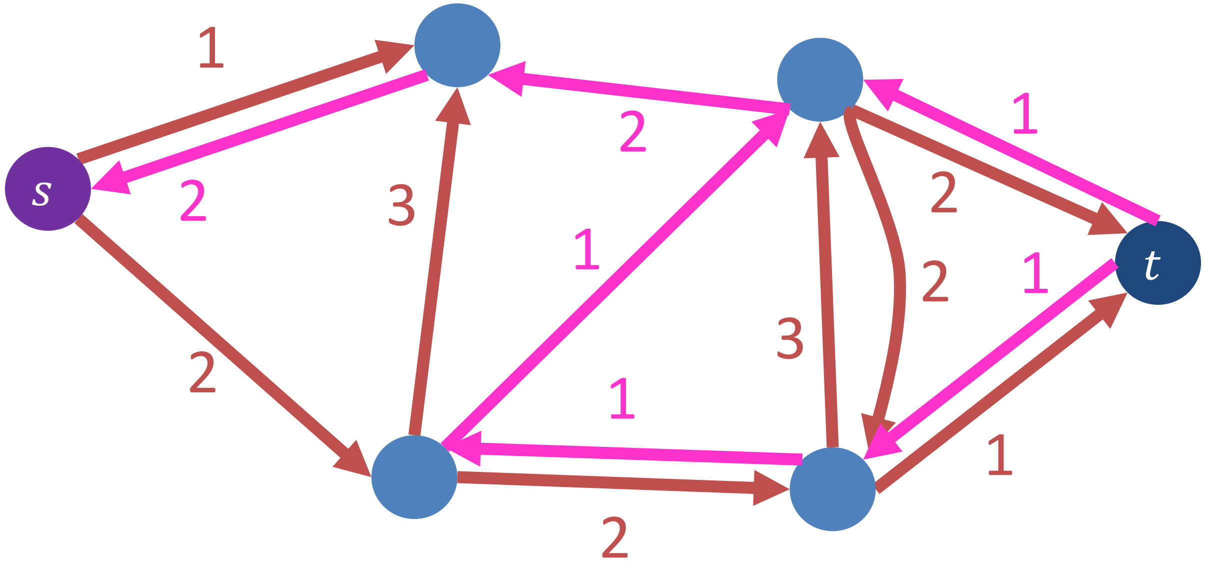

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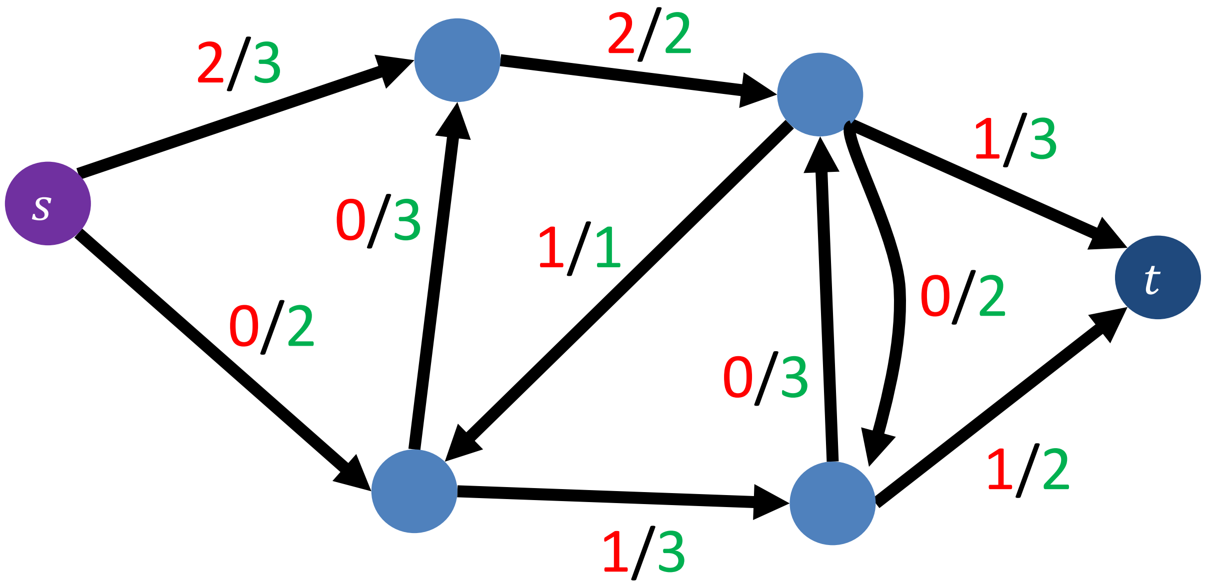
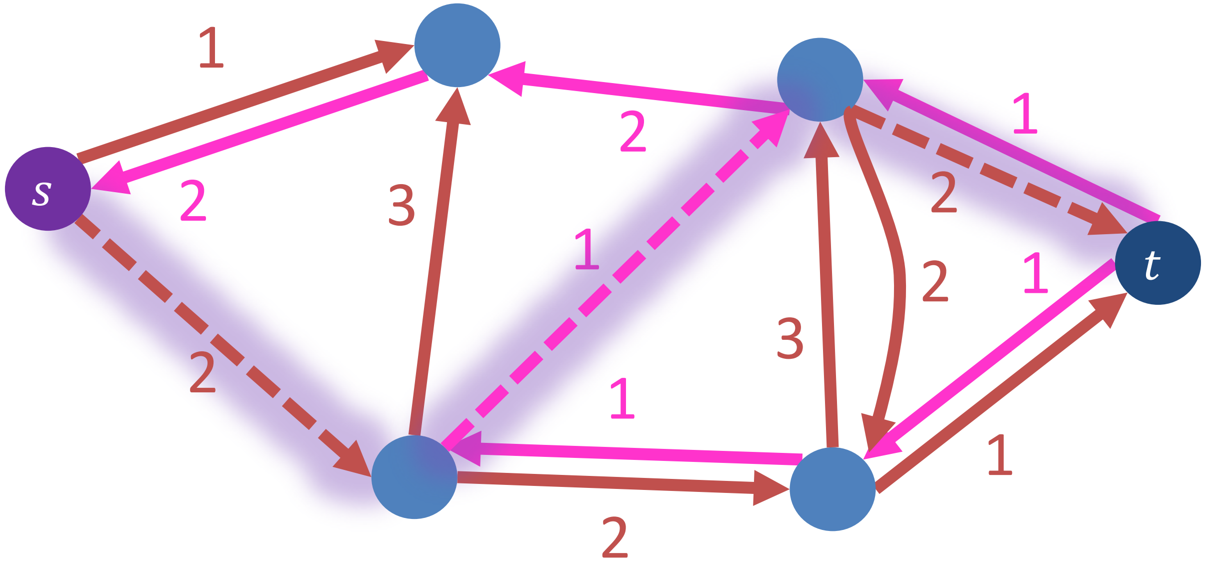
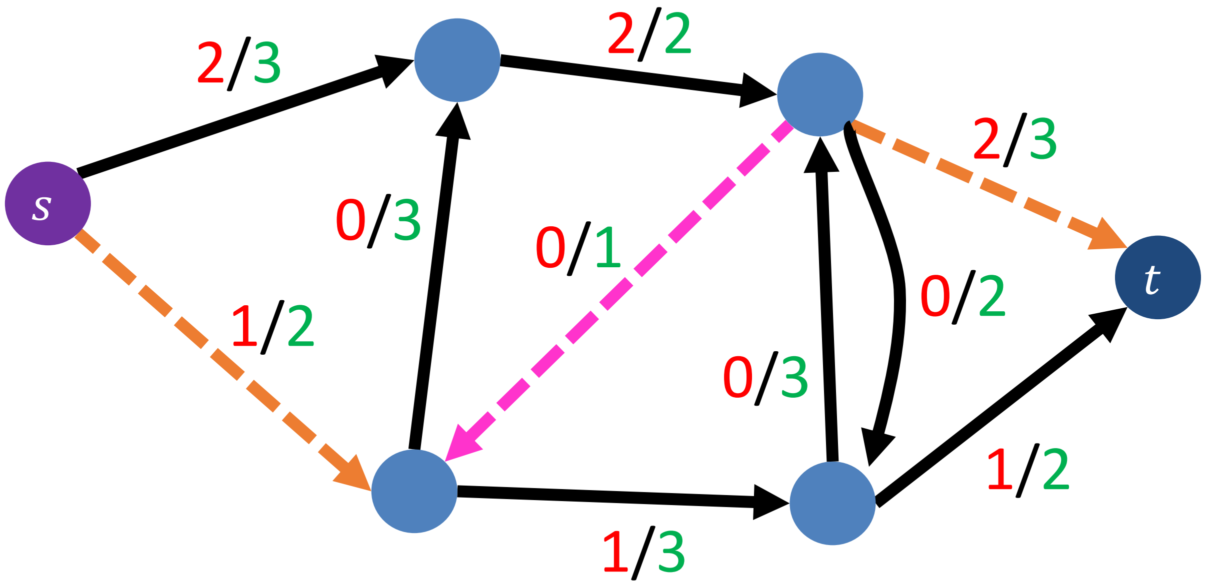




Residual graph G_f

Ford-Fulkerson Example



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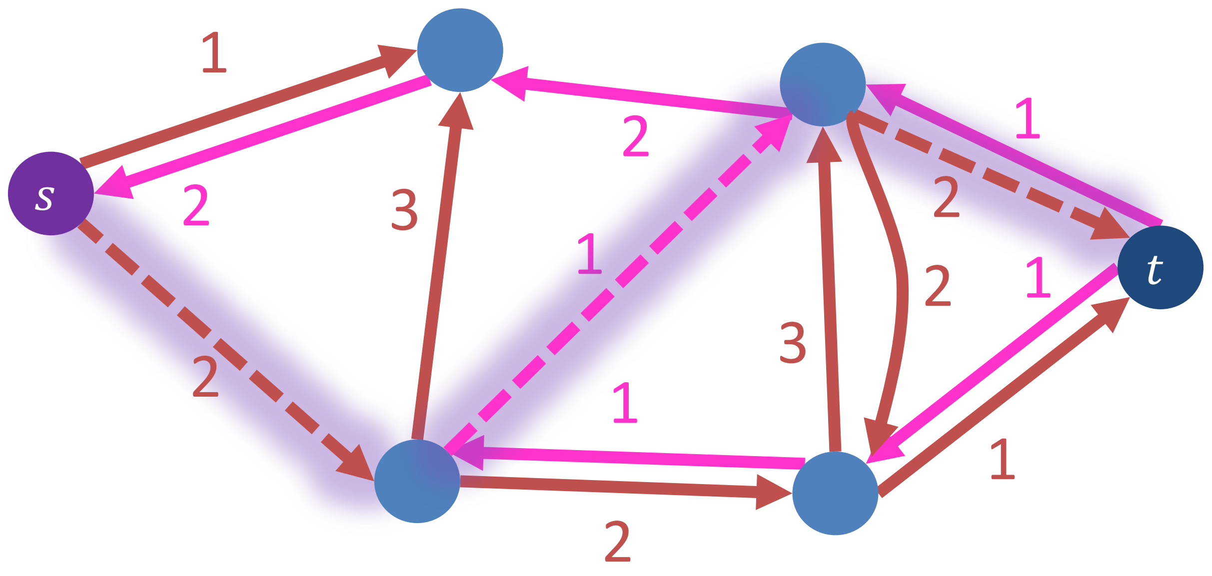
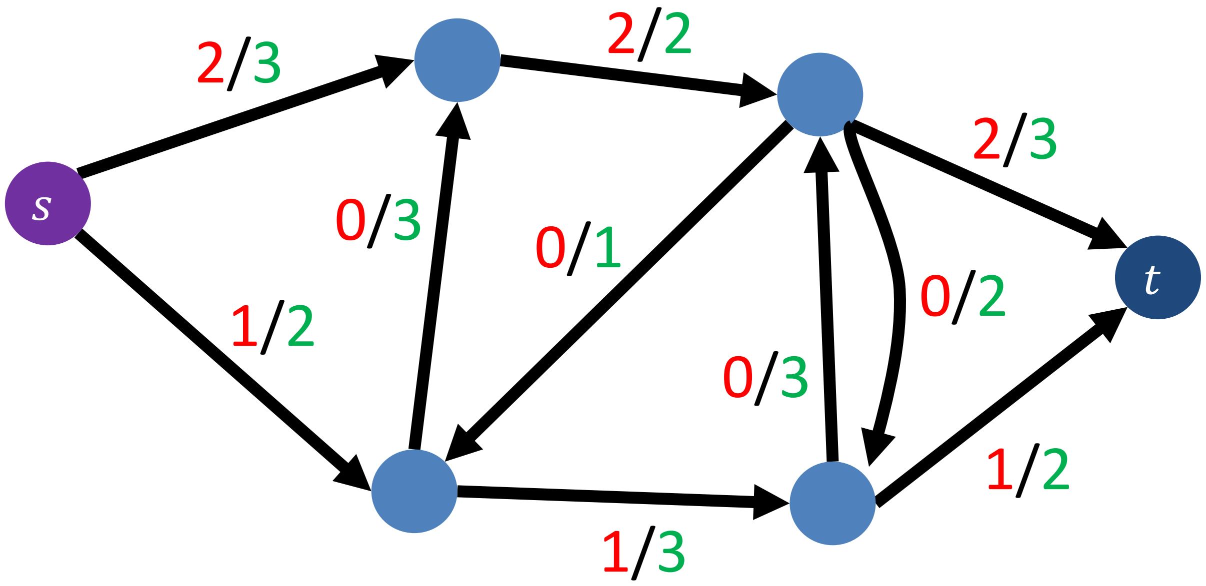


Residual graph G_f

Ford-Fulkerson Example



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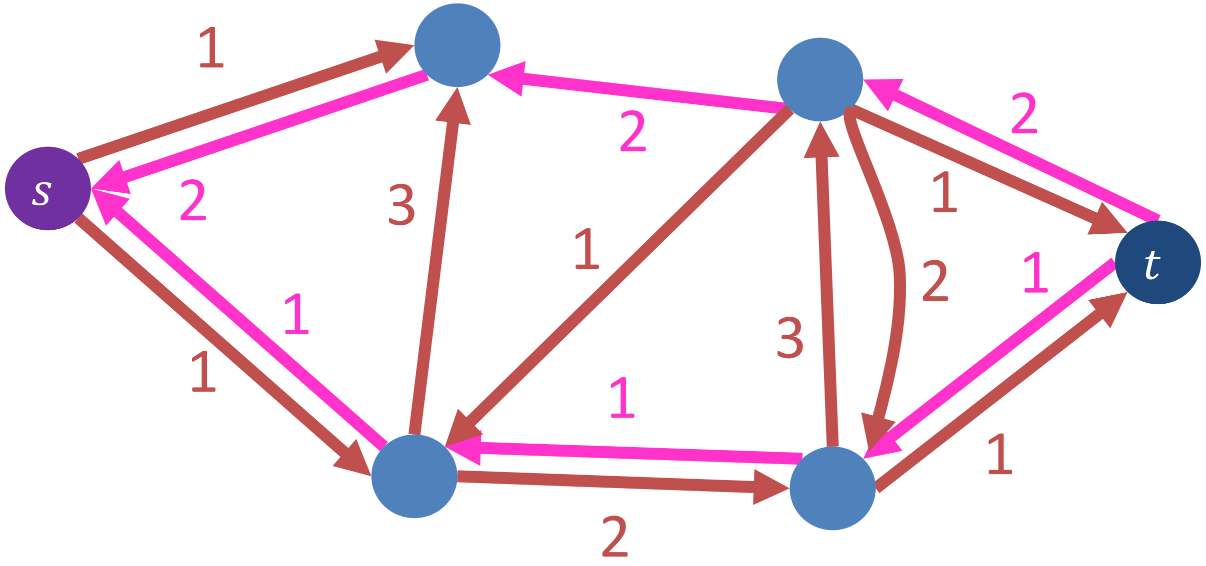

Residual graph G_f

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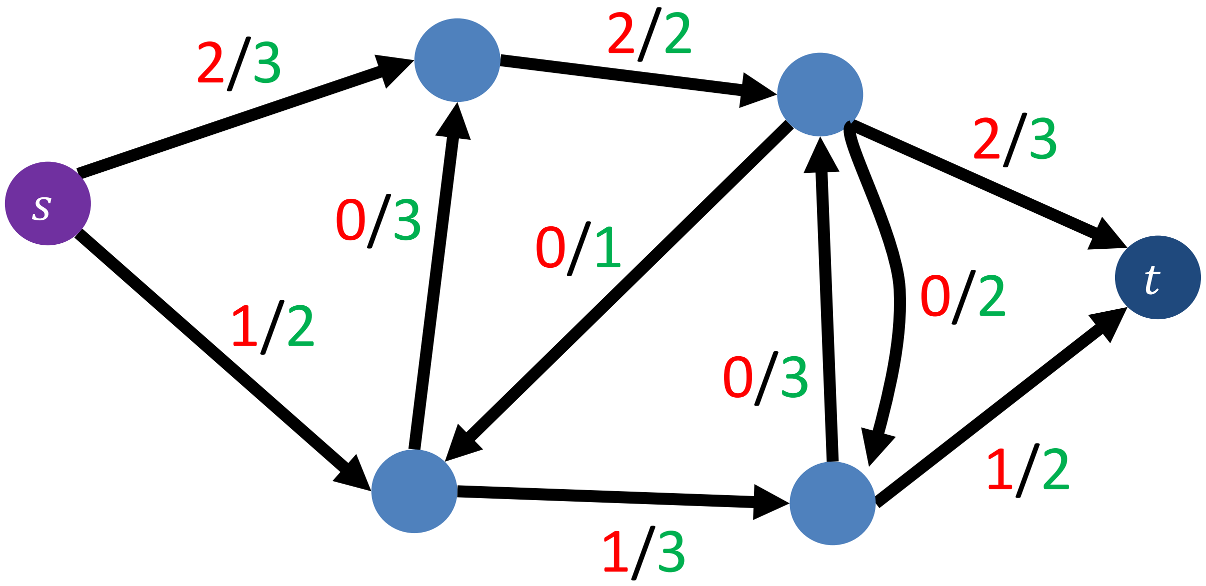
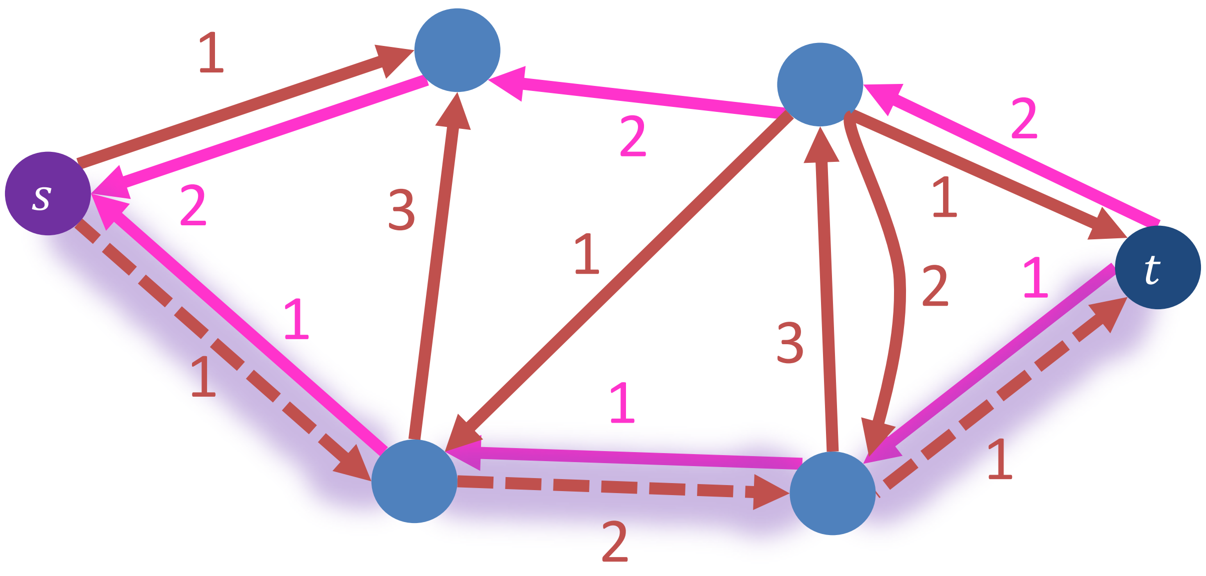
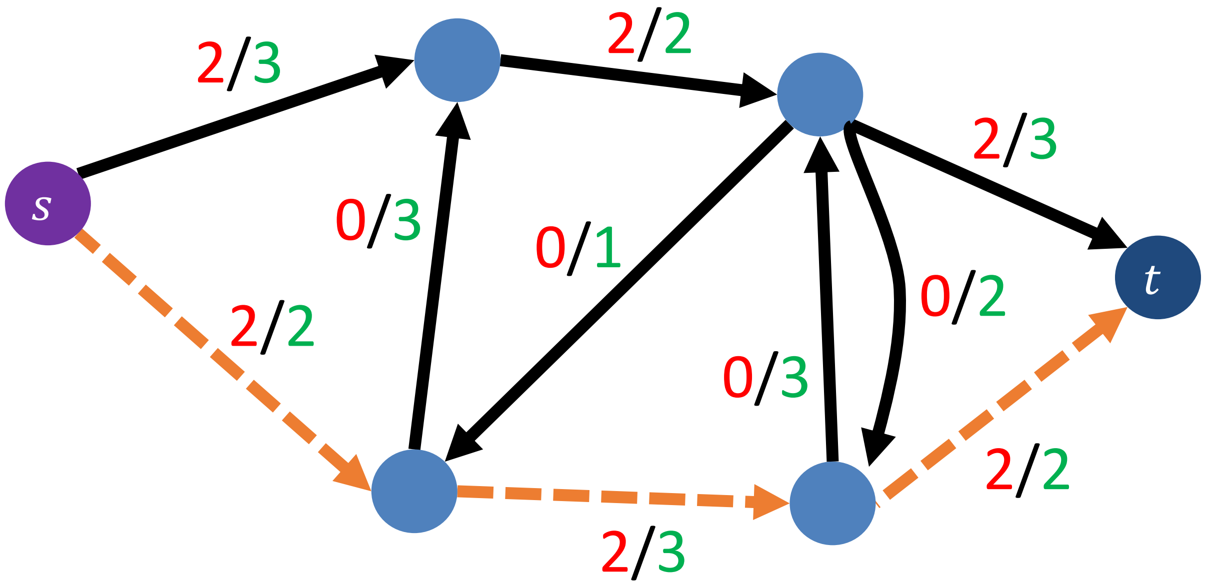




Residual graph G_f

Ford-Fulkerson Example



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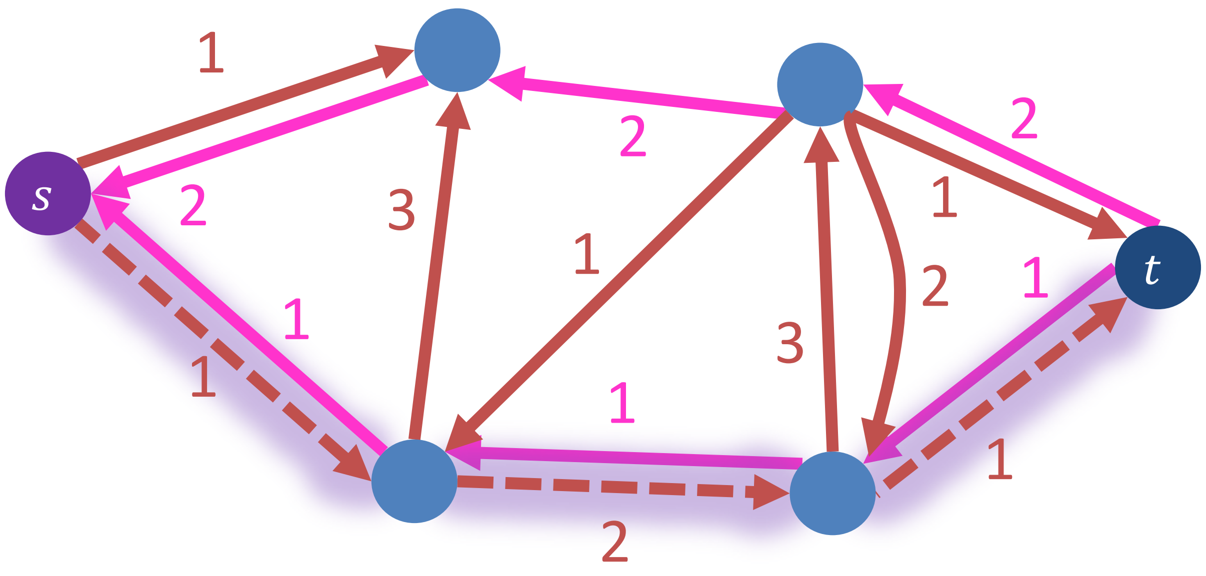

Residual graph G_f

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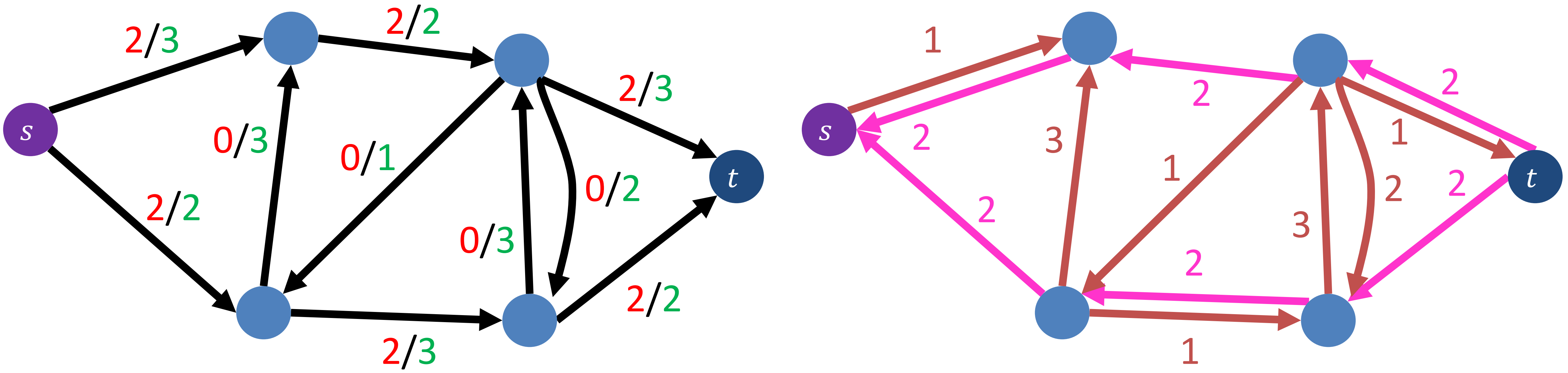
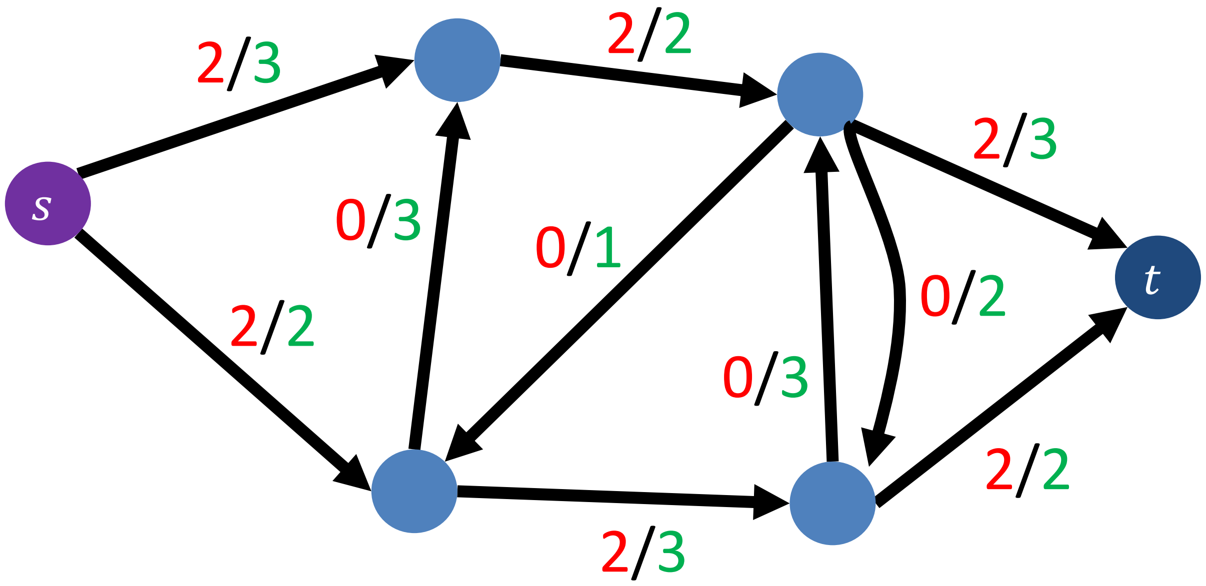
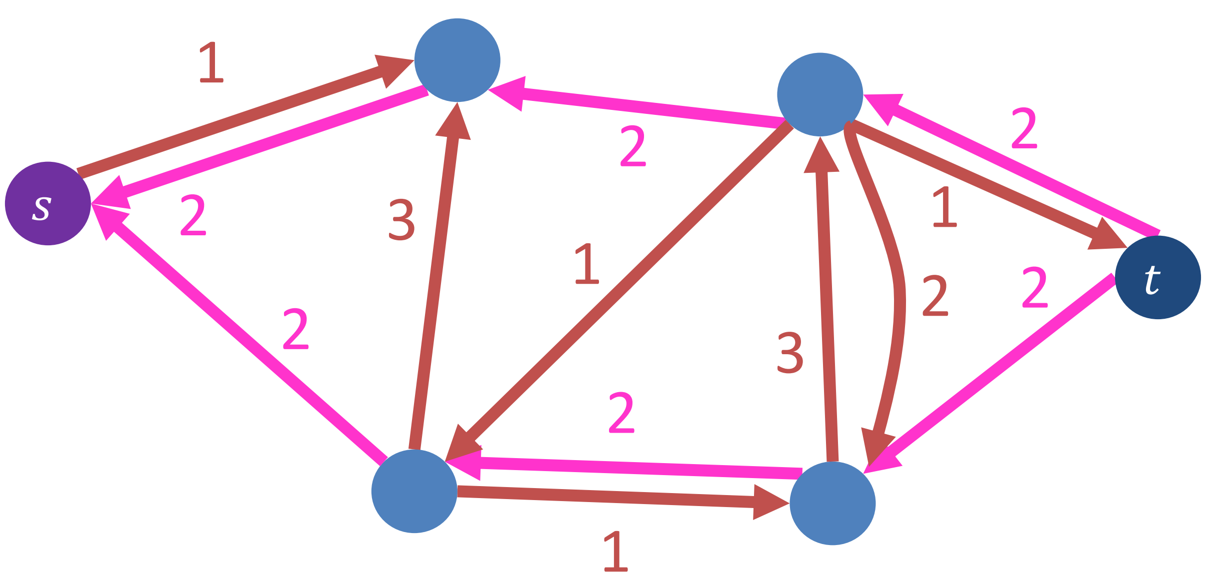


Residual graph G_f

Ford-Fulkerson Example

No more augmenting paths

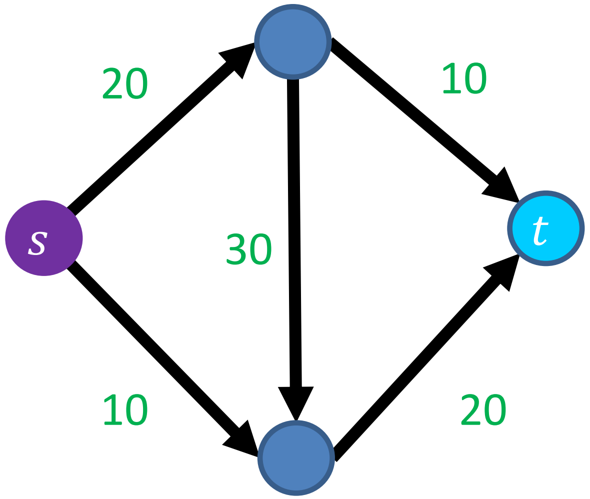




Residual graph G_f

Maximum flow: 4

Our example



Ford-Fulkerson Algorithm - Runtime

Define an **augmenting path** to be a path from $s \rightarrow t$ in the residual graph G_f (using edges of non-zero weight)

Overview: Repeatedly add the flow of any augmenting path

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Time to find an augmenting path:

Number of iterations of While loop:

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Time to find an augmenting path: DFS: $\Theta(V + E)$

Number of iterations of While loop: $|f|$

$\Theta(E \cdot |f|)$

What type of search?

- "While there is an augmenting path p in G_f"
  - Using a depth-first search is the Ford-Fulkerson algorithm
    - Each augmenting path can be found in O(E) time
    - And there can be $|f|$ paths
    - So the running time is O($E \cdot |f|$)
    - Will not terminate with irrational edge values
  - Using a breadth-first search is the Edmonds-Karp algorithm
    - Runs in $O(V \cdot E^2)$
      - Total number of augmentations is O(V·E)
      - And finding each augmentation takes O(E)
    - Guaranteed termination with irrational edge values
    - Run-time is independent of the maximum flow of the graph