CS 3100
Data Structures and Algorithms 2
Lecture 17: Matrix Chaining, Seam Carving

Co-instructors: Robbie Hott and Ray Pettit

Spring 2024

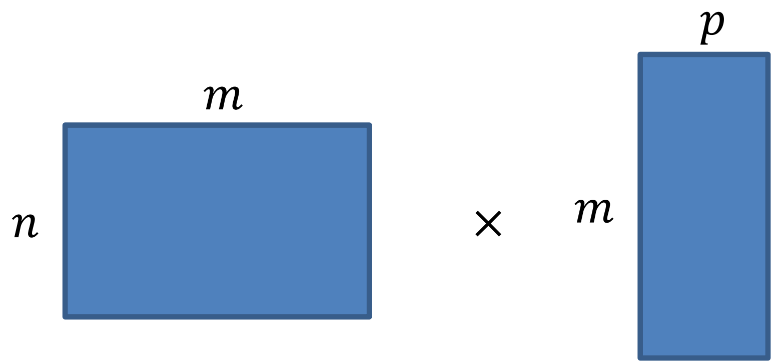
Readings in CLRS 4^{th} edition:

- Chapter 14

Warm Up

How many arithmetic operations are required to multiply a $n \times m$ matrix with a $m \times p$ matrix?

(don't overthink this)



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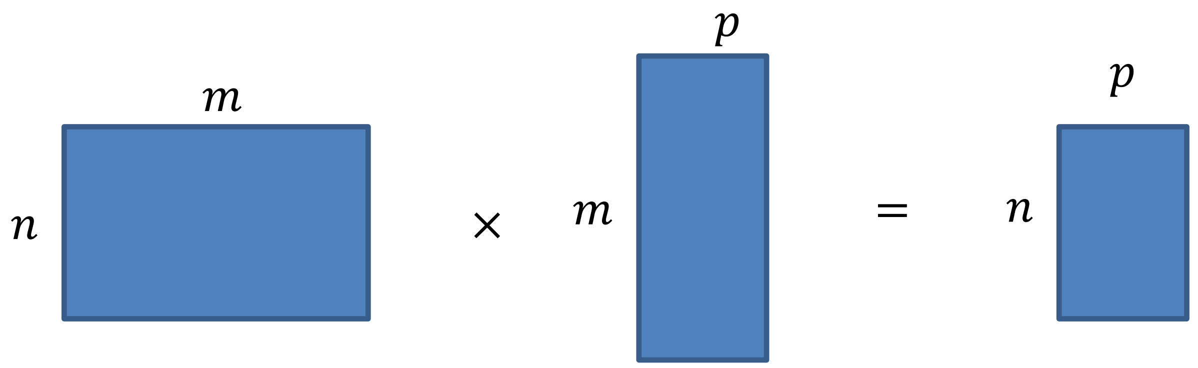

- m multiplications and $m - 1$ additions per element
- $n \cdot p$ elements to compute
- Total cost: $O(m \cdot n \cdot p)$

Announcements

- PS7 due tomorrow
- PA4 now available!
- Office hours
  - Prof Hott Office Hours: Mondays 11a-12p, Fridays 10-11a and 2-3p
  - Prof Pettit Office Hours: Mondays and Fridays 2:30-4:00p
  - TA office hours posted on our website
  - Office hours are not for "checking solutions"

Greedy Algorithms

Optimal Solution to big problem

- Require two things:
  - Optimal Substructure
  - Greedy Choice Function
- Optimal Substructure:
  - If A is an optimal solution to a problem, then the components of A are optimal solutions to subproblems
- Greedy Choice Function
  - The rule for how to choose an item guaranteed be in the optimal solution
- Greedy Algorithm Procedure:
  - Apply the Greedy Choice Function to pick an item
  - Identify your subproblem, then solve it

Dynamic Programming

- Requires Optimal Substructure
  - Solution to larger problem contains the (optimal) solutions to smaller ones
- Idea:
  1. Identify the recursive structure of the problem
     - What is the “last thing” done?
  2. Save the solution to each subproblem in memory
  3. Select a good order for solving subproblems
     - “Top Down”: Solve each recursively
     - “Bottom Up”: Iteratively solve smallest to largest

Generic Divide and Conquer Solution

```
def myDCalgo(problem):

    if baseCase(problem):
        solution = solve(problem)

        return solution
    for subproblem of problem:    # After dividing
        subsolutions.append(myDCalgo(subproblem))
    solution = Combine(subsolutions)

    return solution
```

Generic Top-Down Dynamic Programming Soln

```
mem = {}
def myDPalgo(problem):
        if mem[problem] not blank:
                return mem[problem]
        if baseCase(problem):
                solution = solve(problem)
                mem[problem] = solution
                return solution
        for subproblem of problem:
                subsolutions.append(myDPalgo(subproblem))
        solution = OptimalSubstructure(subsolutions)
        mem[problem] = solution
        return solution
```

Log Cutting

Given a log of length n
A list (of length n) of prices P ($P[i]$ is the price of a cut of size i)
Find the best way to cut the log

Price:	1	5	8	9	10	17	17	20	24	30
Length:	1	2	3	4	5	6	7	8	9	10

Select a list of lengths $\ell_1, \ldots, \ell_k$ such that:
$\sum \ell_i = n$
to maximize $\sum P[\ell_i]$

Brute Force: $O(2^n)$

Greedy Algorithm

- Greedy algorithms build a solution by picking the best option "right now"
  - Select the most profitable cut first

Price:	1	18	24	36	50	50
Length:	1	2	3	4	5	6

Greedy: Lengths: 5, 1
Profit: 51

Better: Lengths: 2, 4
Profit: 54

Greedy Algorithm

- Greedy algorithms build a solution by picking the best option "right now"
  - Select the "most bang for your buck"
    - (best price / length ratio)

Price:	1	18	24	36	50	50
Length:	1	2	3	4	5	6

Greedy: Lengths: 5, 1
Profit: 51

Better: Lengths: 2, 4
Profit: 54

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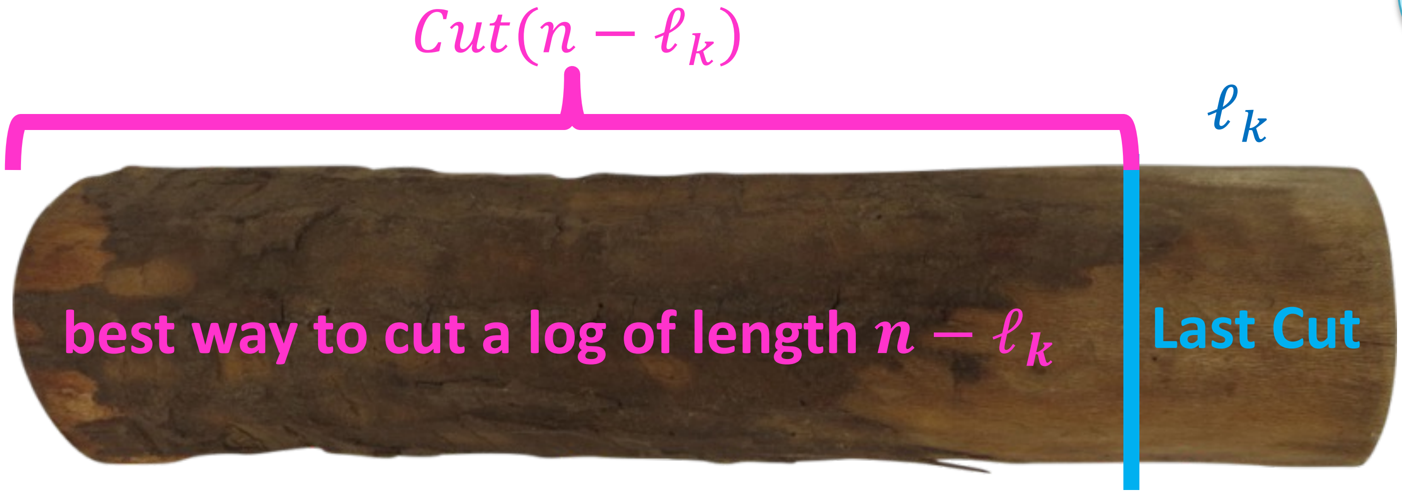
1. Identify Recursive Structure

$P[i]$ = value of a cut of length i

$Cut(n)$ = value of best way to cut a log of length n

$$Cut(n) = \max \begin{cases} Cut(n-1) + P[1] \\ Cut(n-2) + P[2] \\ \dots \\ Cut(0) + P[n] \end{cases}$$

2. Save sub-solutions to memory!



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3. Select a Good Order for Solving Subproblems

Solve Smallest subproblem first

$$Cut(4) = \max \begin{cases} Cut(3) + P[1] \\ Cut(2) + P[2] \\ Cut(1) + P[3] \\ Cut(0) + P[4] \end{cases}$$

$Cut(i)$:	0										
Length:	0	1	2	3	4	5	6	7	8	9	10

4

Log Cutting Pseudocode

```
Initialize Memory C
Cut(n):
    C[0] = 0
    for i=1 to n:  // log size
        best = 0
        for j = 1 to i: // last cut
            best = max(best, C[i-j] + P[j])
        C[i] = best
    return C[n]
```

Run Time: $O(n^2)$

How to find the cuts?

- This procedure told us the profit, but not the cuts themselves
- Idea: remember the choice that you made, then backtrack

Remember the choice made

```
Initialize Memory C, Choices
Cut(n):
    C[0] = 0
    for i=1 to n:
        best = 0
        for j = 1 to i:
            if best < C[i-j] + P[j]:
                best = C[i-j] + P[j]
                Choices[i]=j
        C[i] = best
    return C[n]
```

Gives the size of the last cut

Reconstruct the Cuts

- Backtrack through the choices

Choices:	0	1	1	2	4	3	4	1	2	4	3
Length:	0	1	2	3	4	5	6	7	8	9	10

1 2 6 7

Example to demo Choices[] only. Profit of 20 is not optimal!

Backtracking Pseudocode

```
i = n
while i > 0:
        print Choices[i]
        i = i – Choices[i]
```

Our Example: Getting Optimal Solution

i	0	1	2	3	4	5	6	7	8	9	10
C[i]	0	1	5	8	10	13	17	18	22	25	30
Choice[i]	0	1	2	3	2	2	6	1	2	3	10

- If n were 5
    - Best score is 13
    - Cut Choice[n]=2, then cut Choice[n-Choice[n]]= Choice[5-2]= Choice[3]=3
- If n were 7
    - Best score is 18
    - Cut 1, then cut 6

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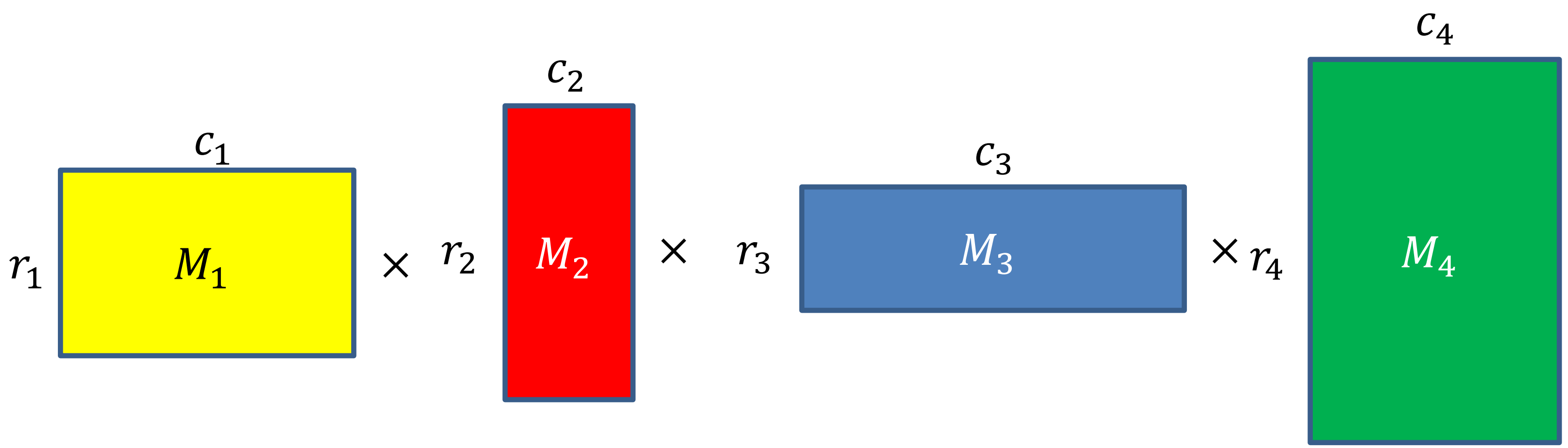
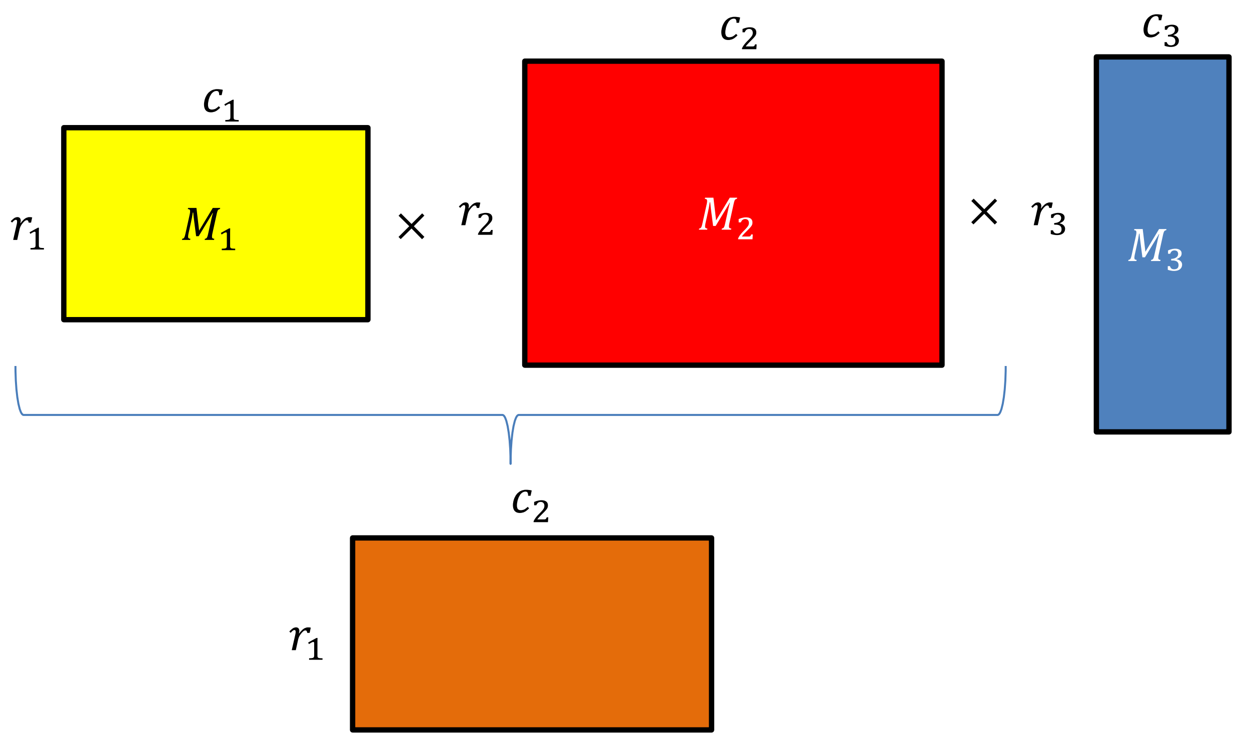
Matrix Chaining

- Given a sequence of Matrices $(M_1, \ldots, M_n)$, what is the most efficient way to multiply them?



Order Matters!

$c_1 = r_2$
$c_2 = r_3$



- $(M_1 \times M_2) \times M_3$
  - uses $(c_1 \cdot r_1 \cdot c_2) + c_2 \cdot r_1 \cdot c_3$ multiplications

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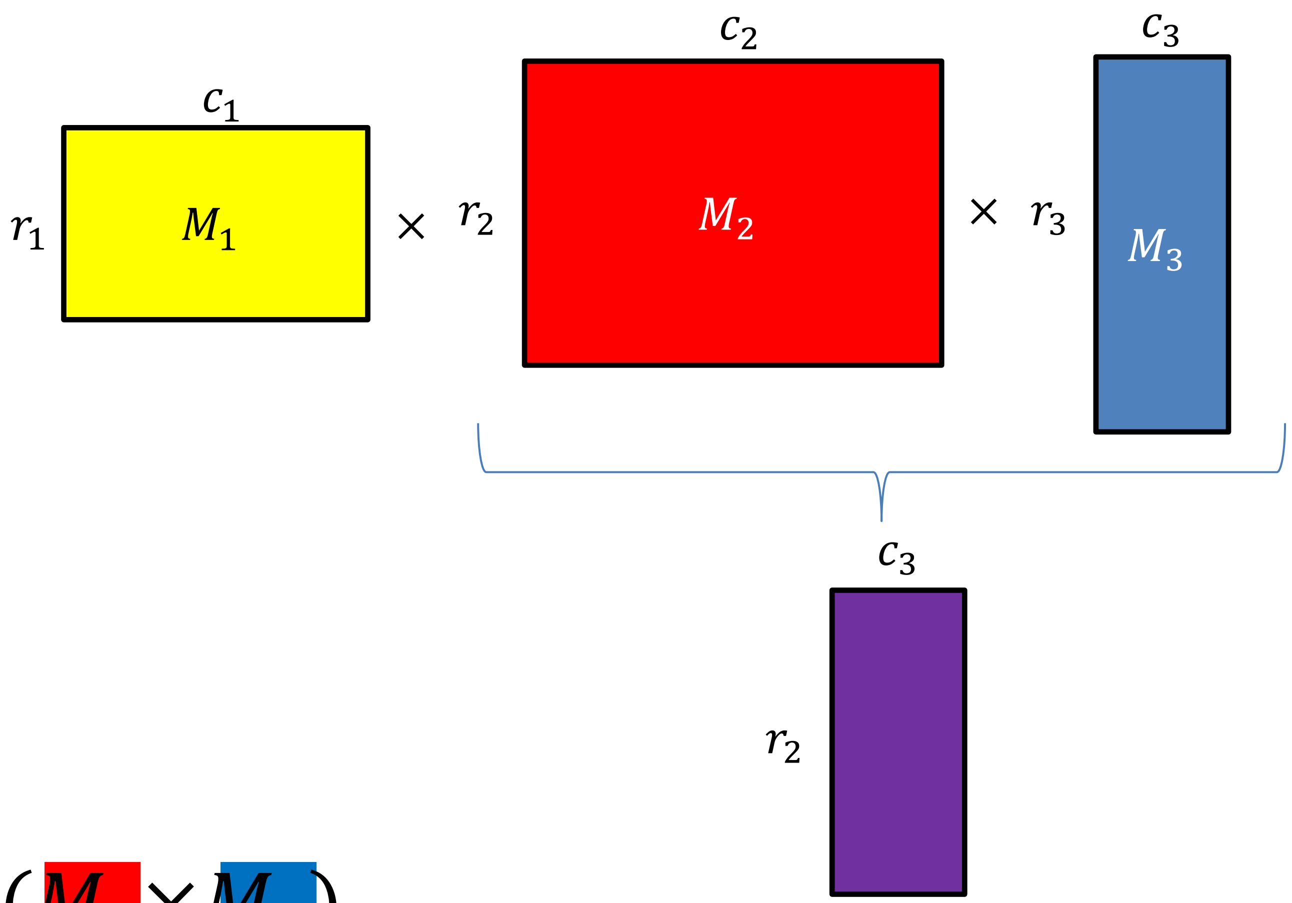

- $M_1 \times (M_2 \times M_3)$
  - uses $c_1 \cdot r_1 \cdot c_3 + (c_2 \cdot r_2 \cdot c_3)$ multiplications

Order Matters!

$c_1 = r_2$
$c_2 = r_3$

- $(M_1 \times M_2) \times M_3$
  - uses $(c_1 \cdot r_1 \cdot c_2) + \mathrm{c}_2 \cdot r_1 \cdot c_3$ multiplications
  - $(10 \cdot 7 \cdot 20) + 20 \cdot 7 \cdot 8 = 2520$
- $M_1 \times (M_2 \times M_3)$
  - uses $c_1 \cdot r_1 \cdot c_3 + (\mathrm{c}_2 \cdot r_2 \cdot c_3)$ multiplications
  - $10 \cdot 7 \cdot 8 + (20 \cdot 10 \cdot 8) = 2160$

$M_1 = 7 \times 10$
$M_2 = 10 \times 20$
$M_3 = 20 \times 8$

$c_1 = 10$
$c_2 = 20$
$c_3 = 8$
$r_1 = 7$
$r_2 = 10$
$r_3 = 20$

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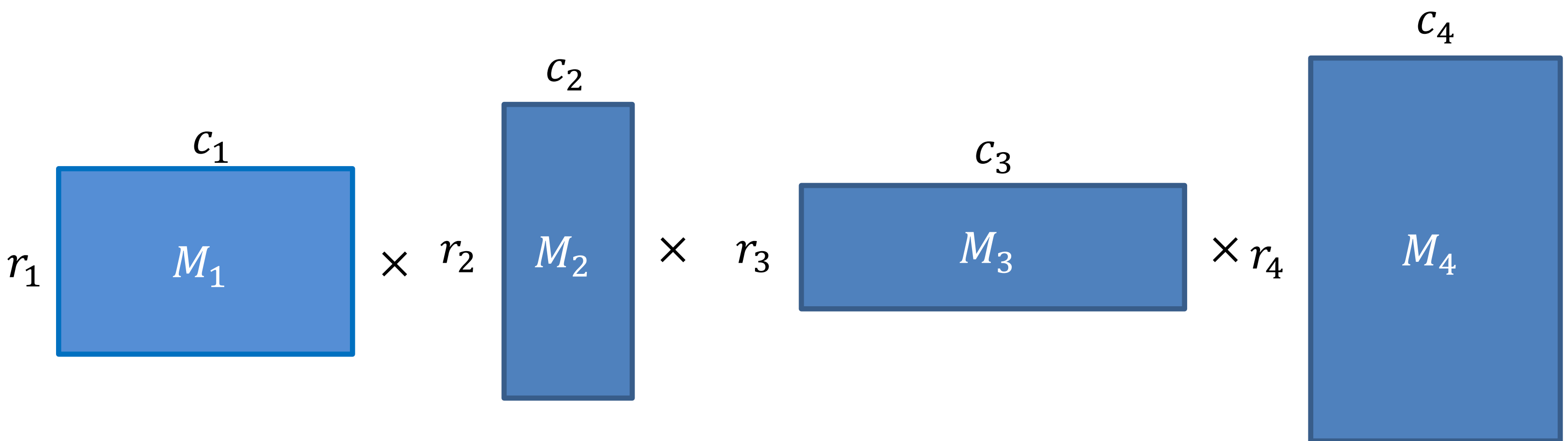
1. Identify the Recursive Structure of the Problem

$Best(1, n)$ = cheapest way to multiply together M_1 through M_n



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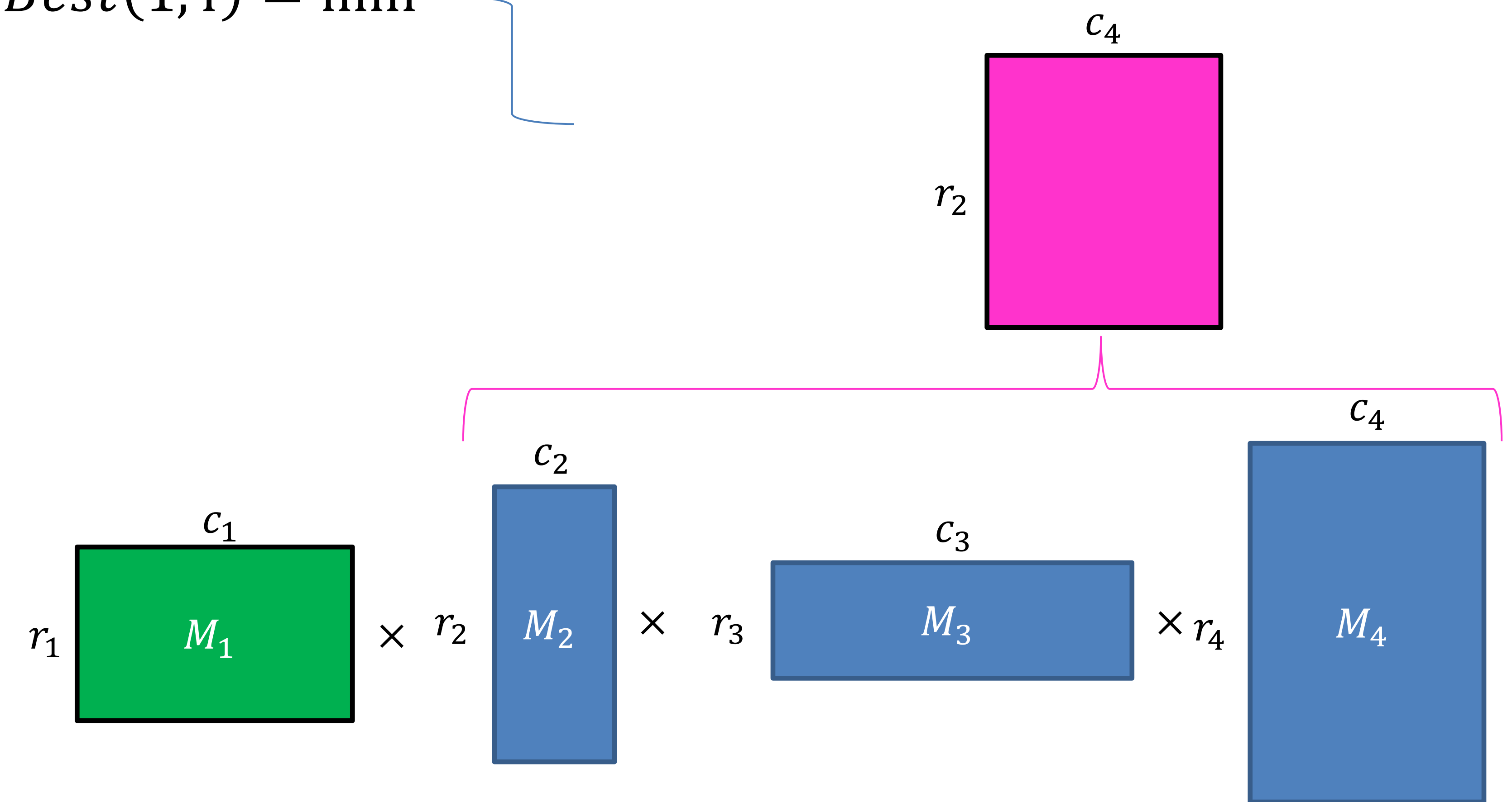

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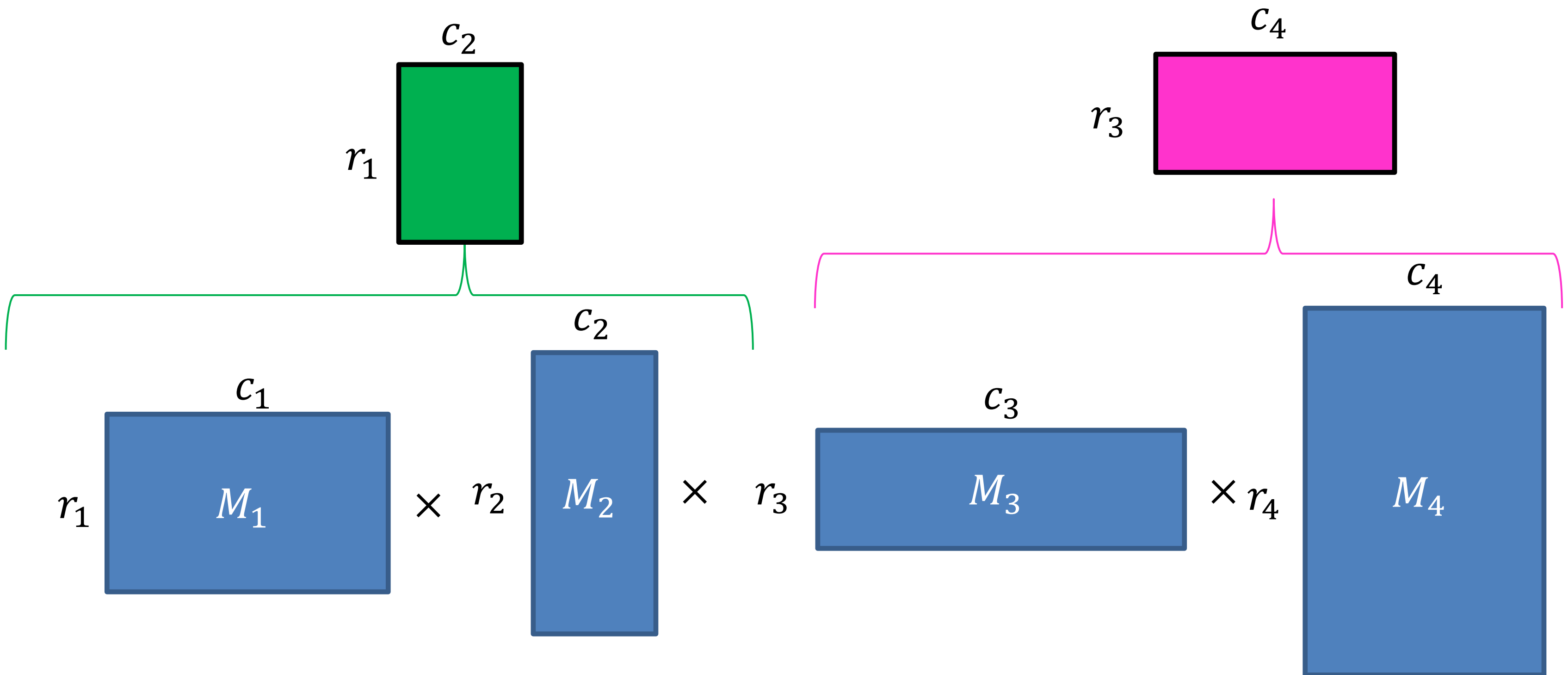

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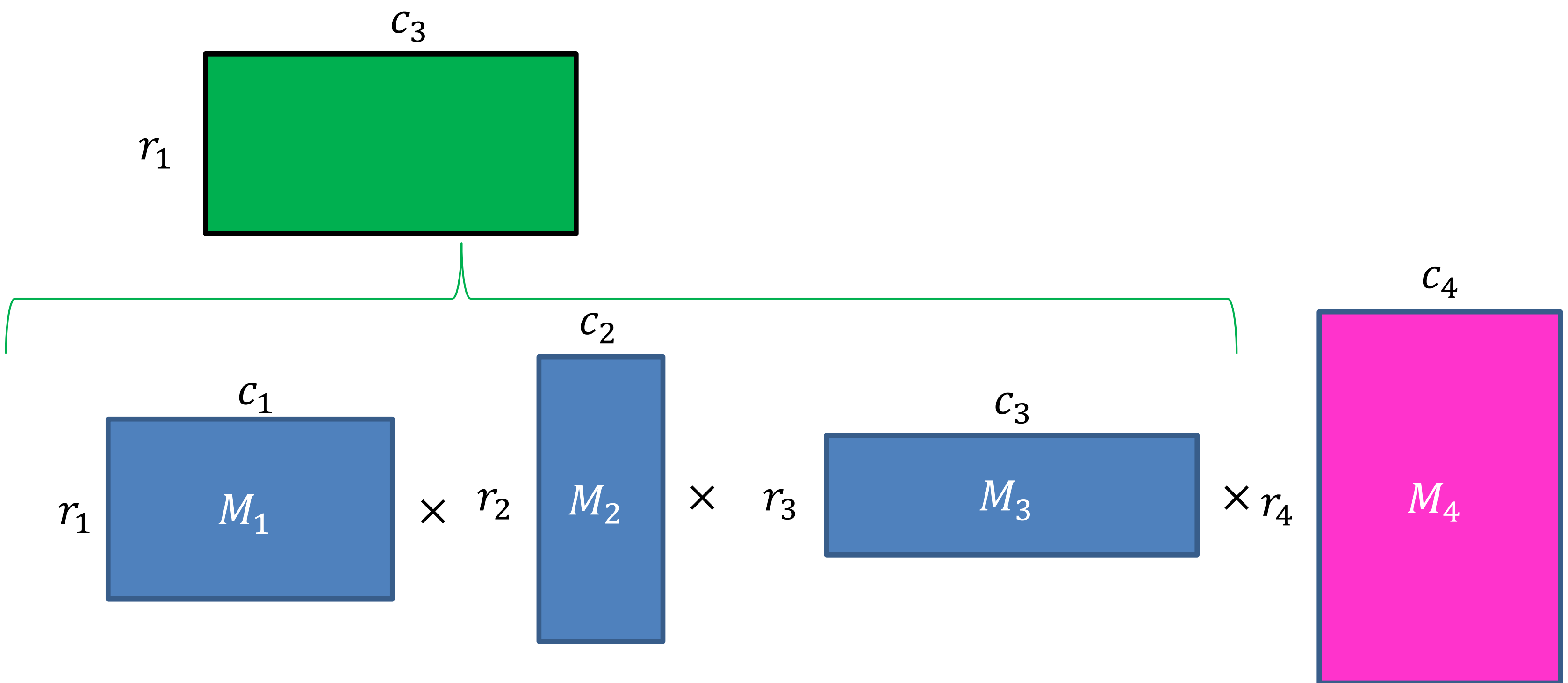

1. Identify the Recursive Structure of the Problem

- In general:

$Best(i,j) =$ cheapest way to multiply together M_i through M_j

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$$Best(1,n) = \min \begin{cases} Best(2,n) + r_1 r_2 c_n \\ Best(1,2) + Best(3,n) + r_1 r_3 c_n \\ Best(1,3) + Best(4,n) + r_1 r_4 c_n \\ Best(1,4) + Best(5,n) + r_1 r_5 c_n \\ \quad \dots \\ Best(1,n-1) + r_1 r_n c_n \end{cases}$$

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$$Best(i,i) = 0$$

Read from M[n]
if present

Save to M[n]

$$Best(1,n) = \min \begin{cases} Best(2,n) + r_1 r_2 c_n \\ Best(1,2) + Best(3,n) + r_1 r_3 c_n \\ Best(1,3) + Best(4,n) + r_1 r_4 c_n \\ Best(1,4) + Best(5,n) + r_1 r_5 c_n \\ \quad \dots \\ Best(1,n-1) + r_1 r_n c_n \end{cases}$$

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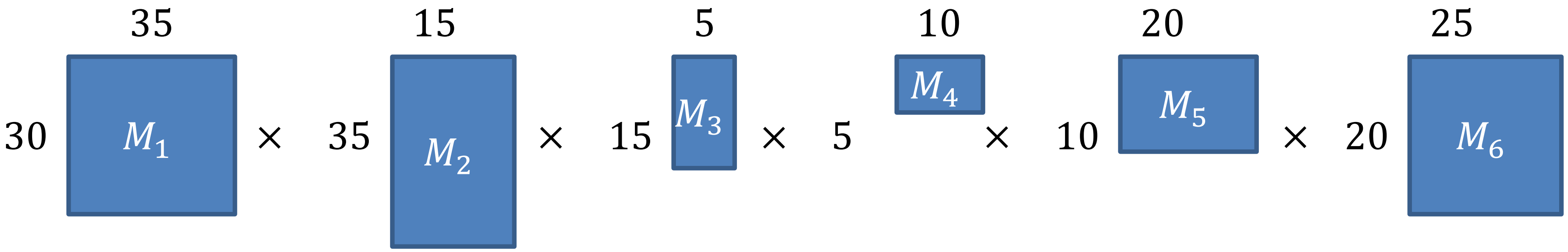
Read from M[n] if present

Save to M[n]

$$Best(1,n) = \min \begin{cases} Best(2,n) + r_1 r_2 c_n \\ Best(1,2) + Best(3,n) + r_1 r_3 c_n \\ Best(1,3) + Best(4,n) + r_1 r_4 c_n \\ Best(1,4) + Best(5,n) + r_1 r_5 c_n \\ \quad \ldots \\ Best(1,n-1) + r_1 r_n c_n \end{cases}$$

3. Select a good order for solving subproblems



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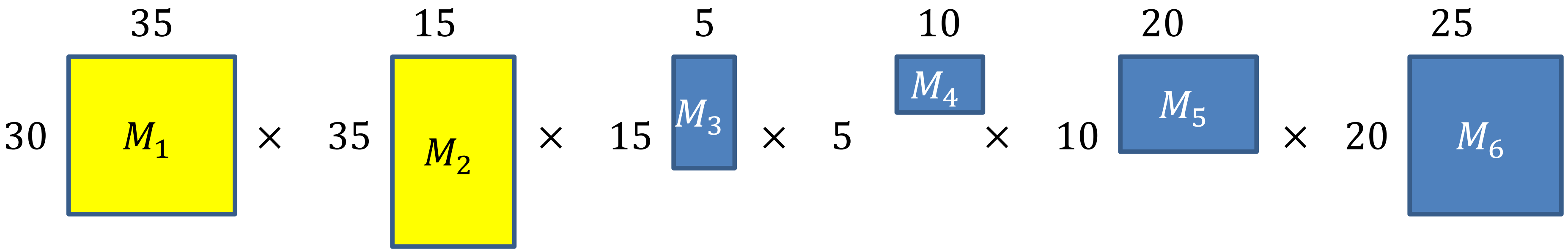
$$Best(i,i) = 0$$

$i =$ \ $j =$	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

3. Select a good order for solving subproblems



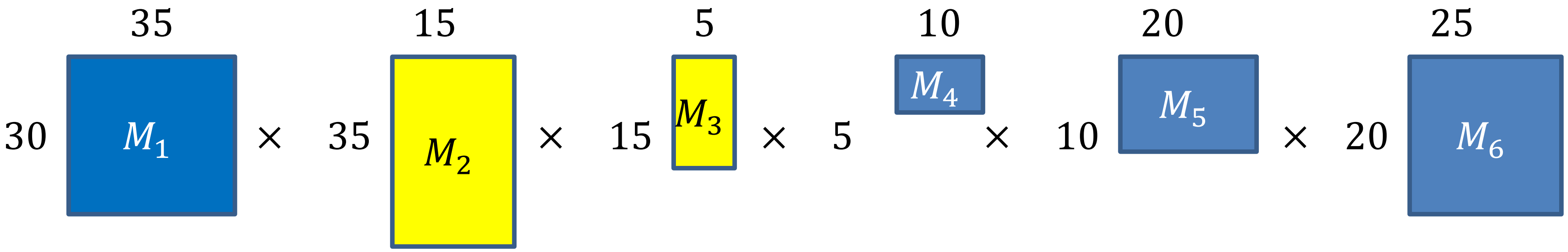
$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$i =$
0	15750					1
	0					2
		0				3
			0			4
				0		5
					0	6

$$Best(1,2) = \min\left\{ Best(1,1) + Best(2,2) + r_1 r_2 c_2 \right.$$

3. Select a good order for solving subproblems



$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$	1	2	3	4	5	6	$i =$
	0	15750					1
		0	2625				2
			0				3
				0			4
					0		5
						0	6

$$Best(2,3) = \min\left\{ Best(2,2) + Best(3,3) + r_2 r_3 c_3 \right.$$

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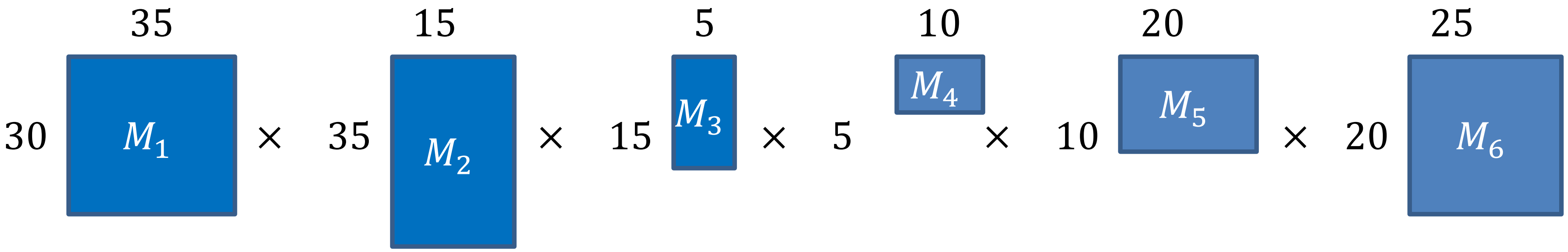
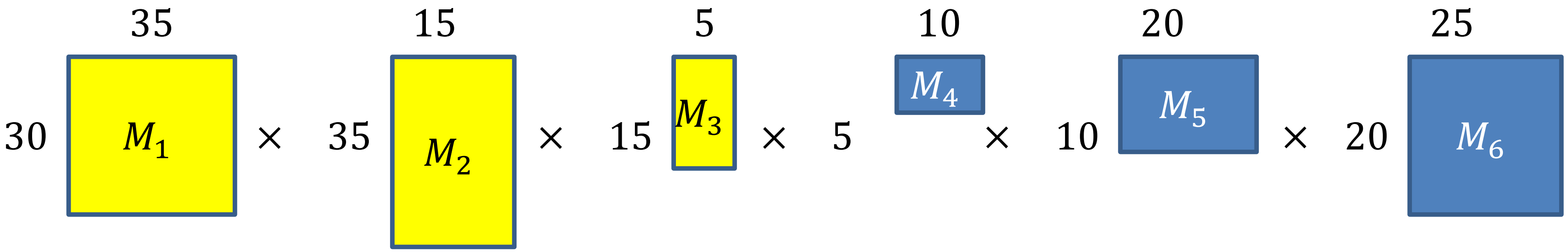


$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$i =$
0	15750					1
	0	2625				2
		0	750			3
			0	1000		4
				0	5000	5
					0	6

3. Select a good order for solving subproblems



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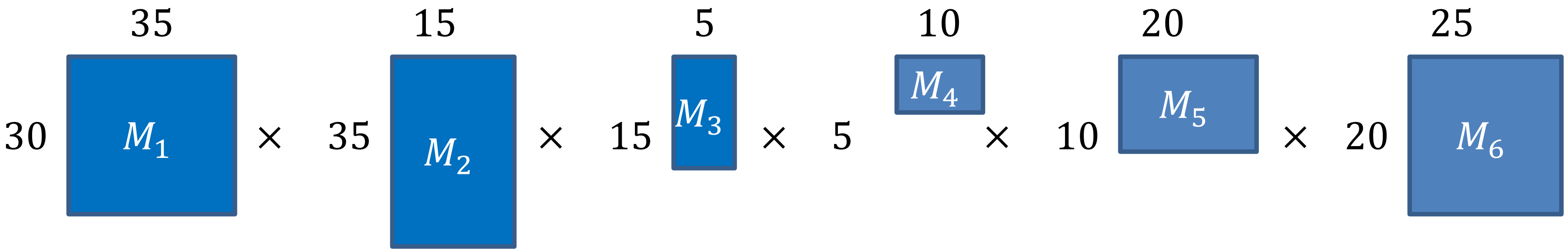
$$Best(i,i) = 0$$

$$r_1 r_2 c_3 = 30 \cdot 35 \cdot 5 = 5250$$
$$r_1 r_3 c_3 = 30 \cdot 15 \cdot 5 = 2250$$

$j =$	1	2	3	4	5	6	$= i$
	0	15750	7875				1
		0	2625				2
			0	750			3
				0	1000		4
					0	5000	5
						0	6

$$Best(1,3) = \min\begin{cases} \overbrace{Best(1,1)}^{0} + \overbrace{Best(2,3)}^{2625} + r_1 r_2 c_3 \\ \underbrace{Best(1,2)}_{15750} + \underbrace{Best(3,3)}_{0} + r_1 r_3 c_3 \end{cases}$$

3. Select a good order for solving subproblems



$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$	1	2	3	4	5	6	$i =$
	0	15750	7875				1
		0	2625				2
			0	750			3
				0	1000		4
					0	5000	5
						0	6

To find $Best(i,j)$: Need all preceding terms of row i and column j

Conclusion: solve in order of diagonal

Matrix Chaining

$$\underset{30 \times 35}{M_1} \times \underset{35 \times 15}{M_2} \times \underset{15 \times 5}{M_3} \times \underset{5 \times 10}{M_4} \times \underset{10 \times 20}{M_5} \times \underset{20 \times 25}{M_6}$$

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$= i$
0	15750	7875	9375	11875	15125	1
	0	2625	4375	7125	10500	2
		0	750	2500	5375	3
			0	1000	3500	4
				0	5000	5
					0	6

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ Best(1,3) + Best(4,6) + r_1 r_4 c_6 \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

Run Time

1. Initialize $Best[i,i]$ to be all 0s — $\Theta(n^2)$ cells in the Array
2. Starting at the main diagonal, working to the upper-right, fill in each cell using:
   1. $Best[i,i] = 0$
   2. $Best[i,j] = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$

$\Theta(n)$ options for each cell

Each "call" to Best() is a O(1) memory lookup

$\Theta(n^3)$ overall run time

Backtrack to find the best order

"remember" which choice of k was the minimum at each cell

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$= i$
0	15750	7875 1	9375	11875	15125 3	1
	0	2625	4375	7125	10500	2
		0	750	2500	5375	3
			0	1000	3500 5	4
				0	5000	5
					0	6

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ \boxed{Best(1,3) + Best(4,6) + r_1 r_4 c_6} \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

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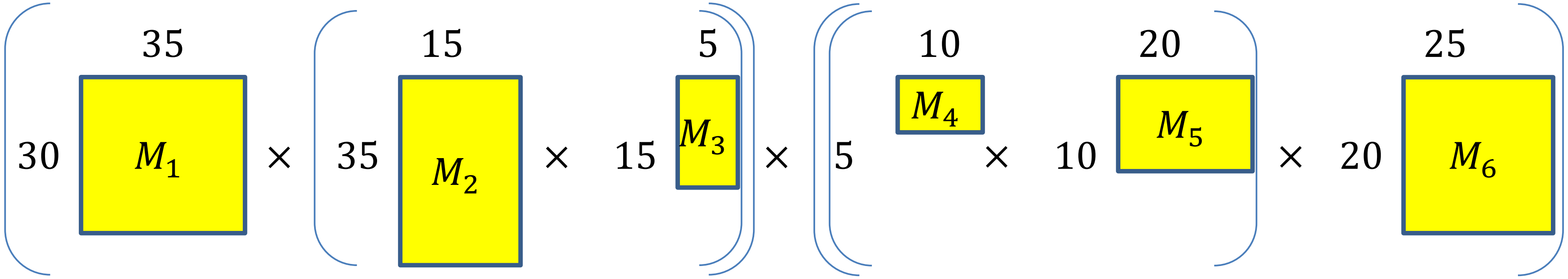
Matrix Chaining



$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

i \ $j =$	1	2	3	4	5	6
1	0	15750	7875 $_{1}$	9375	11875	15125 $_{3}$
2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500 $_{5}$
5					0	5000
6						0

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ \boxed{Best(1,3) + Best(4,6) + r_1 r_4 c_6} \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

Storing and Recovering Optimal Solution

- Maintain table Choice[i,j] in addition to Best table
  - Choice[i,j] = k means the best "split" was right after M_k
  - Work backwards from value for whole problem, Choice[1,n]
  - Note: Choice[i,i+1] = i because there are just 2 matrices
- From our example:
  - Choice[1,6] = 3. So $[M_1\ M_2\ M_3]\ [M_4\ M_5\ M_6]$
  - We then need Choice[1,3] = 1. So $[(M_1)\ (M_2\ M_3)]$
  - Also need Choice[4,6] = 5. So $[(M_4\ M_5)\ M_6]$
  - Overall: $[(M_1)\ (M_2\ M_3)]\ [(M_4\ M_5)\ M_6]$

Dynamic Programming

- Requires Optimal Substructure
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Seinfeld break

Movie Time!

In Season 9 Episode 7 “The Slicer” of the hit 90s TV show *Seinfeld*, George discovers that, years prior, he had a heated argument with his new boss, Mr. Kruger. This argument ended in George throwing Mr. Kruger’s boombox into the ocean. How did George make this discovery?

https://www.youtube.com/watch?v=pSB3HdmLcY4

Seam Carving

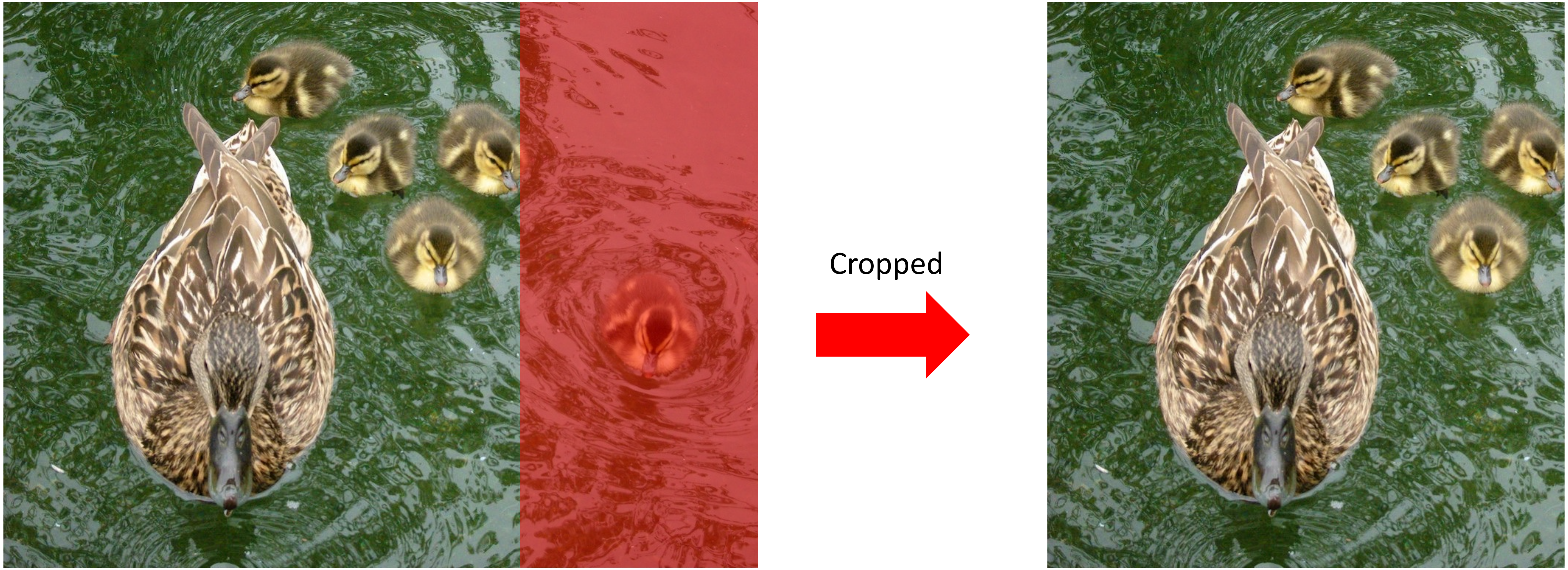
- Method for image resizing that doesn’t scale/crop the image

Seam Carving

- Method for image resizing that doesn't scale/crop the image

Cropping

- Removes a “block” of pixels

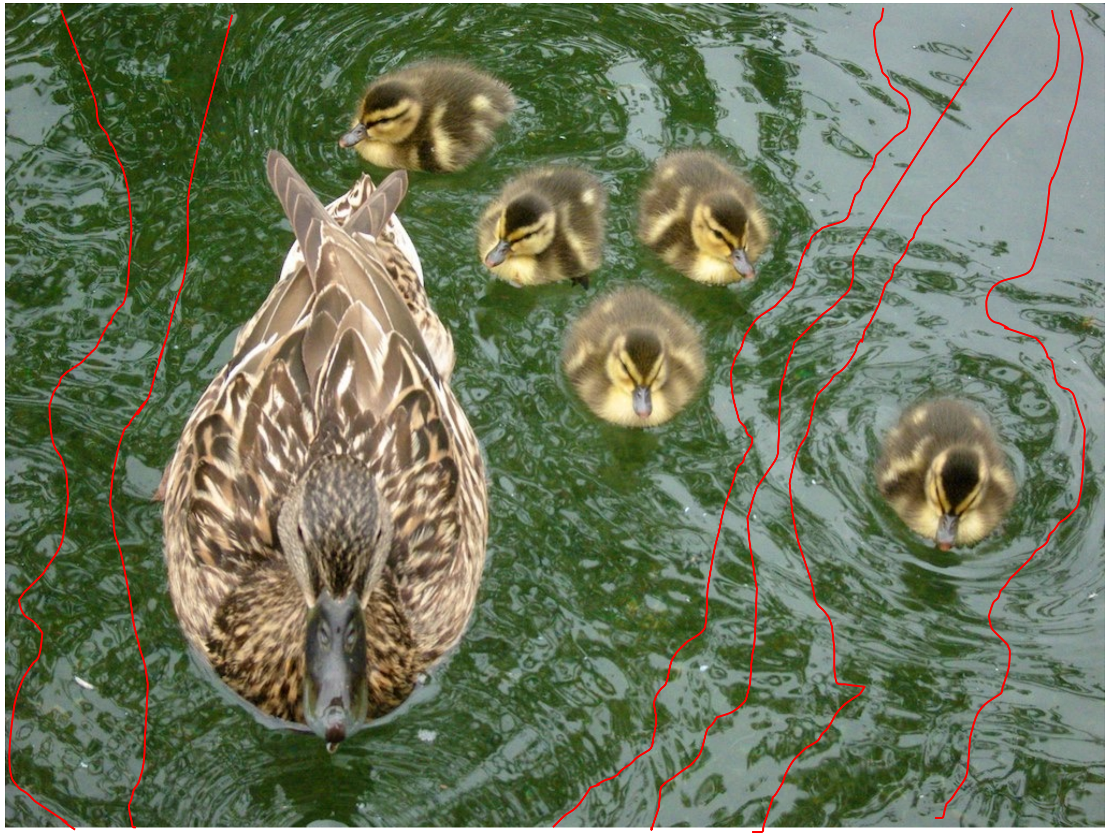


Scaling

- Removes “stripes” of pixels

Seam Carving

- Removes "least energy seam" of pixels
- https://trekhleb.dev/js-image-carver/



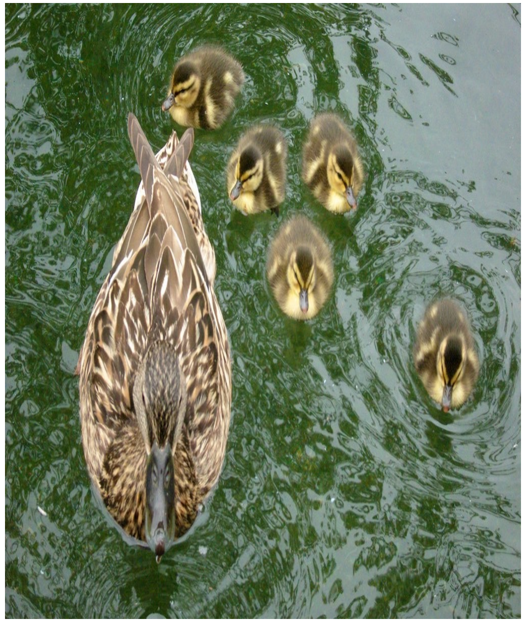
Carved

Seam Carving

- Method for image resizing that doesn't scale/crop the image

Cropped

Scaled



Carved

Seattle Skyline

Energy of a Seam

- Sum of the energies of each pixel

$$e(p) = \text{energy of pixel } p$$

- Many choices for pixel energy
  - E.g.: change of gradient (how much the color of this pixel differs from its neighbors)
  - Particular choice doesn't matter, we use it as a "black box"

- Goal: find least-energy seam to remove

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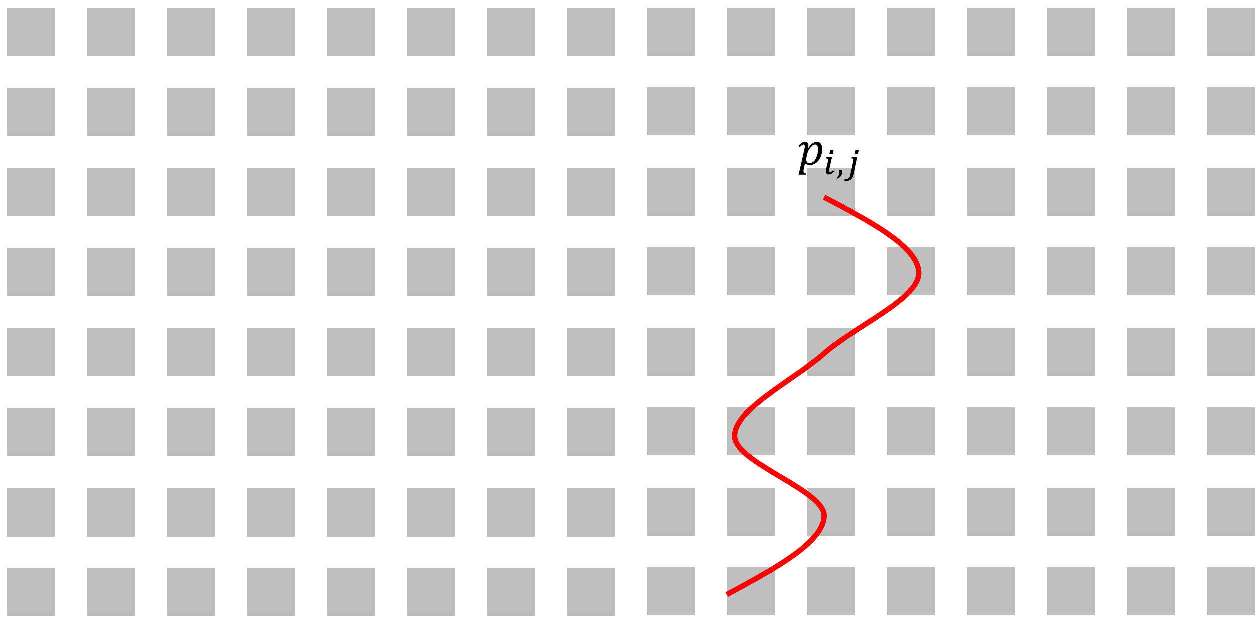
Identify Recursive Structure

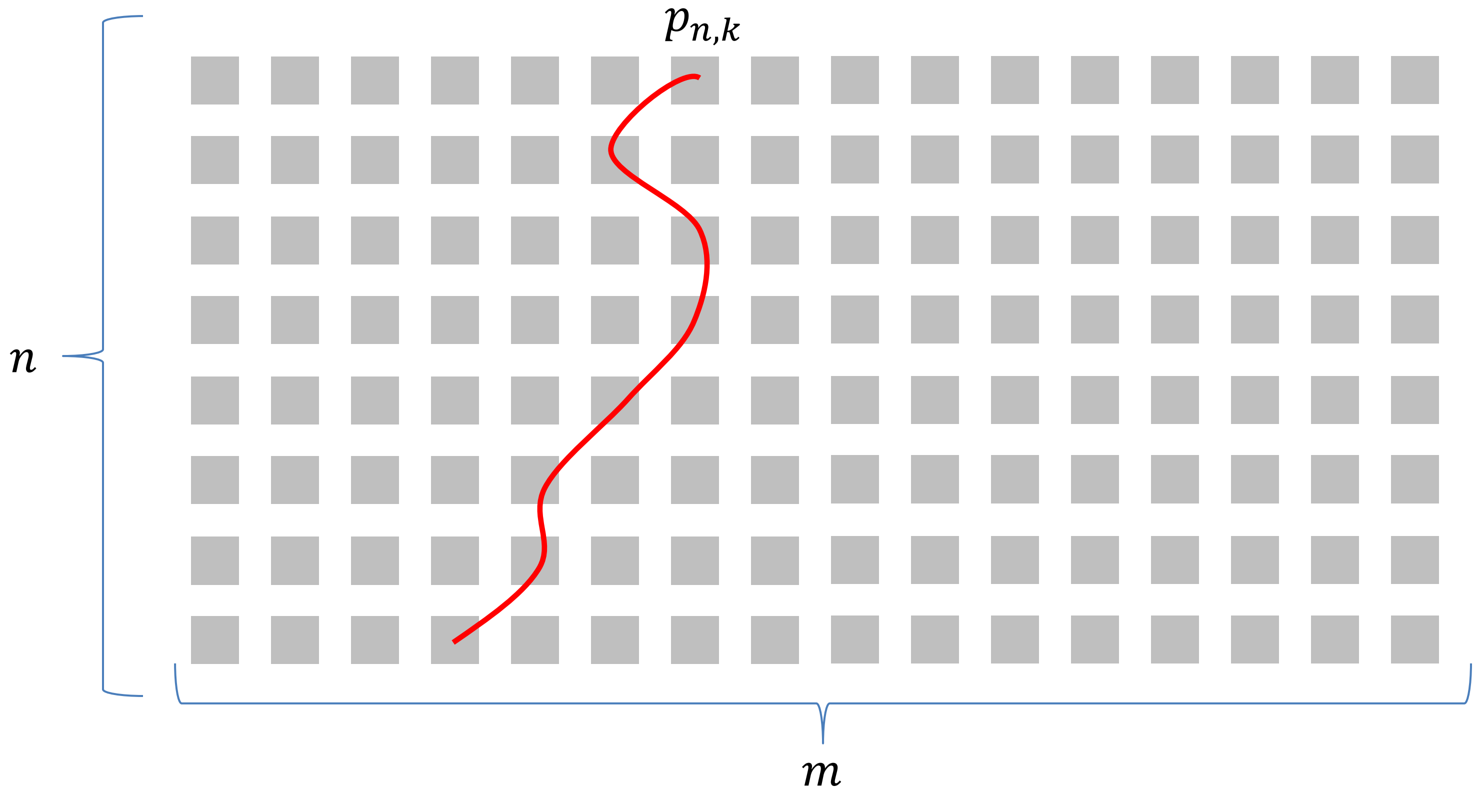
Let $S(i,j)$ = least energy seam from the bottom of the image up to pixel $p_{i,j}$



Finding the Least Energy Seam

Want to delete the least energy seam going from bottom to top, so delete:

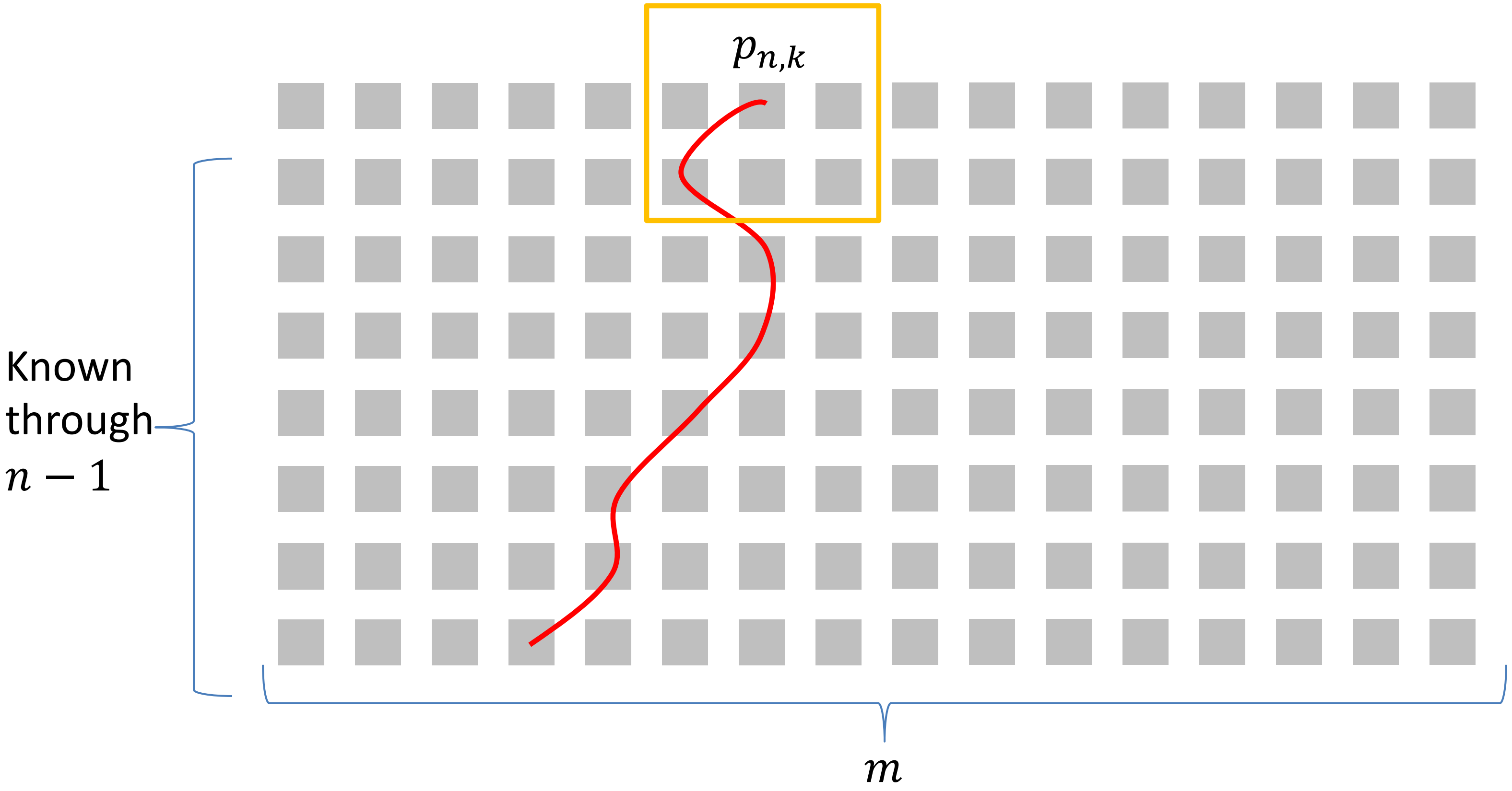
$$\min_{k=1}^{m}(S(n,k))$$



Computing $S(n, k)$

Assume we know the least energy seams for all of row $n - 1$

(i.e. we know $S(n - 1, \ell)$ for all ℓ)



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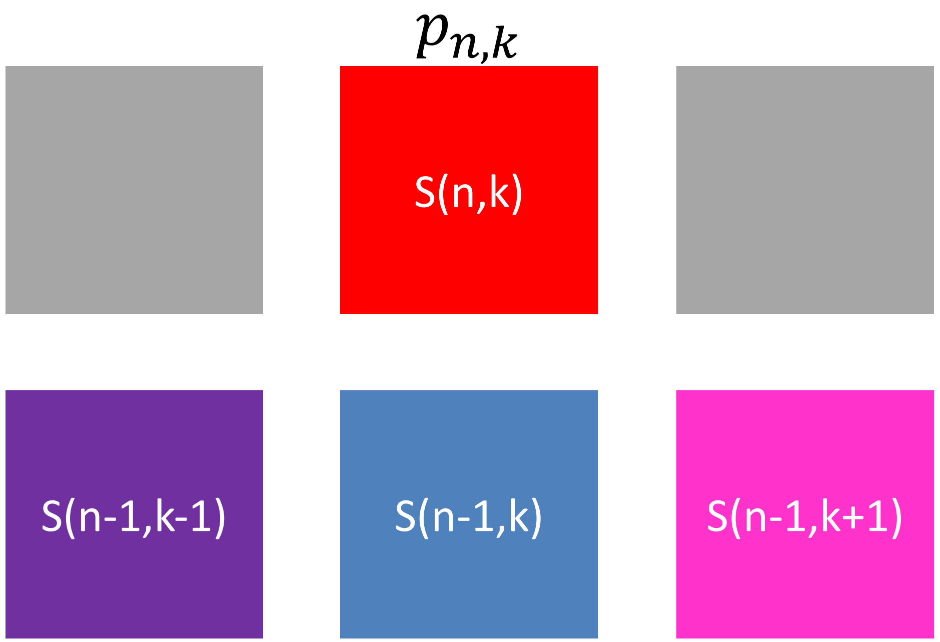
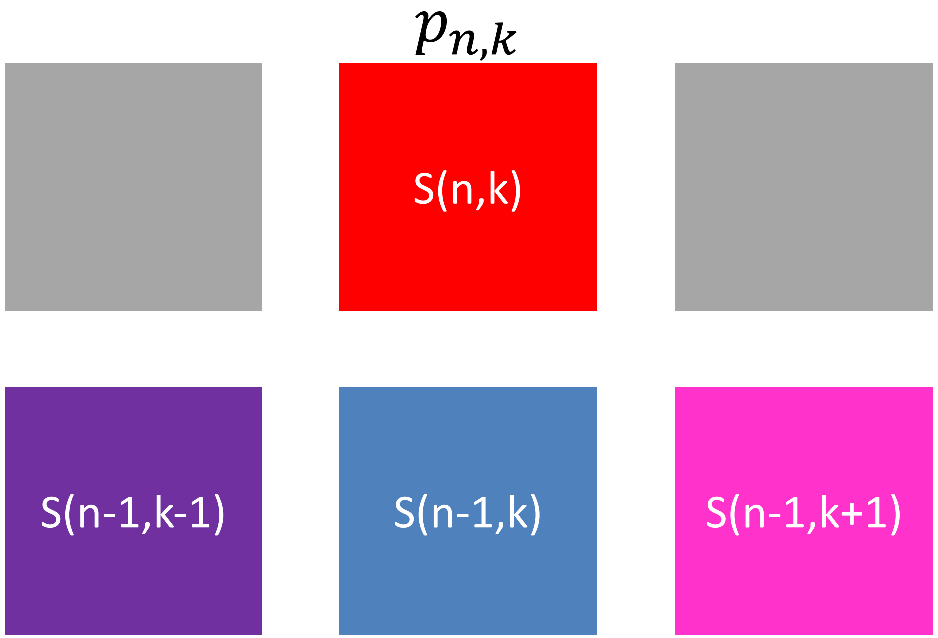
Assume we know the least energy seams for all of row $n-1$ (i.e. we know $S(n-1,\ell)$ for all ℓ)



Computing $S(n,k)$

Assume we know the least energy seams for all of row $n-1$ (i.e. we know $S(n-1,\ell)$ for all ℓ)

$$S(n,k) = min\begin{cases} S(n-1,k-1) + e(p_{n,k}) \\ S(n-1,k) + e(p_{n,k}) \\ S(n-1,k+1) + e(p_{n,k}) \end{cases}$$



Dynamic Programming

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PA4!

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PA4!

Longest Common Subsequence

Given two sequences X and Y, find the length of their longest common subsequence

Example:
$X = ATCTGAT$
$Y = TGCATA$
$LCS = TCTA$

Brute force: Compare every subsequence of X with Y
$\Omega(2^n)$

Dynamic Programming

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1. Identify Recursive Structure

Let $LCS(i,j)$ = length of the LCS for the first i characters of X, first j character of Y

Find $LCS(i,j)$:

Case 1: $X[i] = Y[j]$

$X = ATCTGCGT$

$Y = TGCATAT$

$LCS(i,j) = LCS(i-1, j-1) + 1$

Case 2: $X[i] \neq Y[j]$

$X=ATCTGCGA$

$Y=TGCATAT$

$LCS(i,j) = LCS(i, j-1)$

$X=ATCTGCGT$

$Y=TGCATAC$

$LCS(i,j) = LCS(i-1, j)$

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

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$LCS(i,j) = LCS(i, j-1)$

$X=ATCTGCGT$
$Y=TGCATAC$
$LCS(i,j) = LCS(i-1, j)$

Read from M[i,j] if present

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

Save to M[i,j]

```
X = "alkjdflaksjdf"
Y = "lakjsdflkasjdlfs"
M = 2d array of len(X) rows and len(Y) columns, initialized to -1
def LCS(int i, int j):
        # returns the length of the LCS shared between the length-i prefix of  X and length-j prefix of Y
        # memoization
        if M[i,j] > -1:
                return M[i,j]
        #base case:
        if i == 0 or j == 0:
                ans = 0
        elif X[i] == Y[j]:
                ans = LCS(i-1, j-1) + 1
        else:
                ans = max( LCS(i, j-1), LCS(i-1, j) )
        M[i,j] = ans
        return ans
print(LCS(len(X)+1, len(Y)+1)) # the answer for the entirety of X and Y
```

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

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3. Solve in a Good Order

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$ \ $X =$	0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
T 1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
G 2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
C 3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
A 4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
T 5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
A 6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

To fill in cell (i,j) we need cells $(i-1,j-1),(i-1,j),(i,j-1)$
Fill from Top->Bottom, Left->Right (with any preference)

Run Time?

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$ \ $X =$	0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
0	0	0	0	0	0	0	0	0
T 1	0	0	1	1	1	1	1	1
G 2	0	0	1	1	1	2	2	2
C 3	0	0	1	2	2	2	2	2
A 4	0	1	1	2	2	2	3	3
T 5	0	1	2	2	3	3	3	4
A 6	0	1	2	2	3	3	4	4

Run Time: $\Theta(n \cdot m)$ (for $|X| = n$, $|Y| = m$)

Reconstructing the LCS

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$	$X =$	0	A 1	T 2	C 3	$\boxed{T}$ 4	$\boxed{G}$ 5	$\boxed{A}$ 6	$\boxed{T}$ 7
	0	0	0	0	0	0	0	0	0
$\boxed{T}$	1	0	0	1	1	1	1	1	1
$\boxed{G}$	2	0	0	1	1	1	2	2	2
C	3	0	0	1	2	2	2	2	2
$\boxed{A}$	4	0	1	1	2	2	2	3	3
$\boxed{T}$	5	0	1	2	2	3	3	3	4
A	6	0	1	2	2	3	3	4	4

Start from bottom right,

if symbols matched, print that symbol then go diagonally

else go to largest adjacent

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$Y =$ \ $X =$	0	A 1	$\boxed{T}$ 2	$\boxed{C}$ 3	T 4	G 5	$\boxed{A}$ 6	$\boxed{T}$ 7
0	0	0	0	0	0	0	0	0
$\boxed{T}$ 1	0	0	1	1	1	1	1	1
G 2	0	0	1	1	1	2	2	2
$\boxed{C}$ 3	0	0	1	2	2	2	2	2
$\boxed{A}$ 4	0	1	1	2	2	2	3	3
$\boxed{T}$ 5	0	1	2	2	3	3	3	4
A 6	0	1	2	2	3	3	4	4

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$Y =$ \ $X =$			A	T	C	T	G	A	T
		0	1	2	3	4	5	6	7
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
T	1	**0**	0	1	1	1	1	1	1
G	2	**0**	0	1	1	1	2	2	2
C	3	**0**	0	1	2	2	2	2	2
A	4	**0**	1	1	2	2	2	3	3
T	5	**0**	1	2	2	3	3	3	4
A	6	**0**	1	2	2	3	3	4	4

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if symbols matched, print that symbol then go diagonally

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