CS 3100
Data Structures and Algorithms 2

Lecture 14: Huffman Encoding

Co-instructors: Robbie Hott and Ray Pettit

Spring 2024

Readings in CLRS 4th edition:

- Chapter 16

Warm Up

Decode the line below into English

(hint: use Google or Wolfram Alpha)

·· ·–·· ·· –·– · ·– ·–·· ––· ––– ·–· ·· – ···· –– ···

Warm Up

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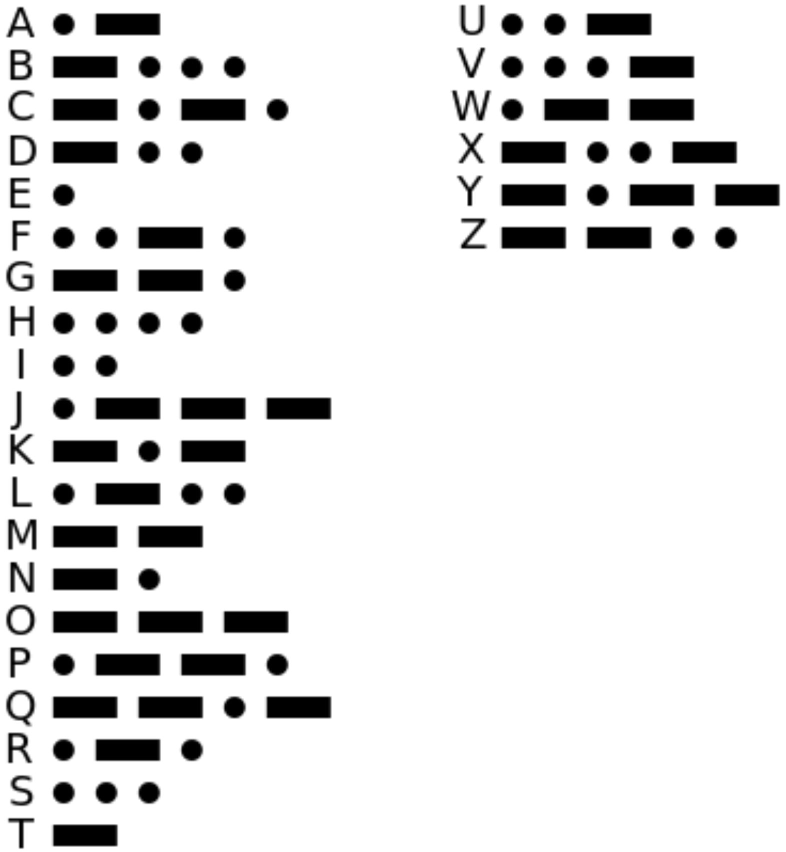
.. .-.. .. -.- . .- .-.. --. --- .-. .. - .... -- ...



Announcements

- PS6 and PA3 coming soon
- Office hours
  - Prof Hott Office Hours: Mondays 11a-12p, Fridays 10-11a and 2-3p
  - Prof Pettit Office Hours: Mondays and Fridays 2:30-4:00p
  - TA office hours posted on our website
  - Office hours are not for "checking solutions"

Reminders about Greedy Algorithms

Reminder: Some Terminology

Optimization problems: terminology

- A solution must meet certain constraints:
  A solution is *feasible*

Example: A possible shortest path must meet these criteria:
All edges must be in the graph and form a simple path.

- Solutions judged on some criteria:

  Objective function

Example: The sum of edge weights in path is minimum

- One (or more) feasible solutions that scores highest (by the objective function) is called the *optimal solution(s)*

The **greedy approach** is often a good choice for optimization problems

- So is **dynamic programming** (coming later in the course)

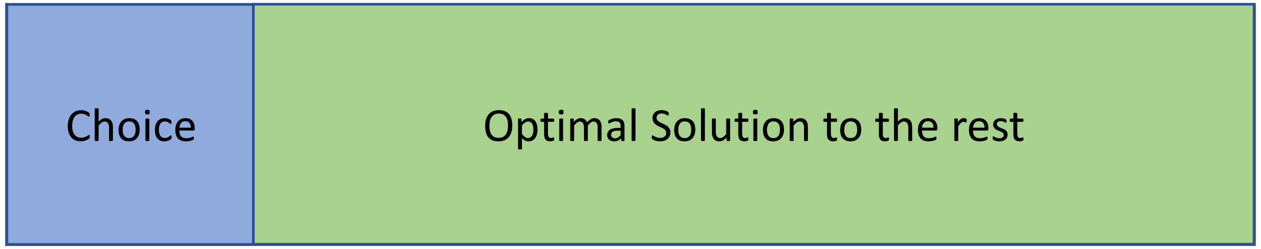
Reminder: Greedy Strategy: An Overview

Greedy strategy:

- Build solution by stages, adding one item to the partial solution we've found before this stage
- At each stage, make *locally optimal choice* based on the ***greedy choice*** (sometimes called the *greedy rule* or the *selection function)*
  - Locally optimal, i.e. best given what info we have now
- Irrevocable: a choice can't be un-done
- Sequence of locally optimal choices leads to globally optimal solution (hopefully)
  - Must prove this for a given problem!

Greedy Algorithms

Optimal Solution to big problem



Require two things:

- Optimal Substructure
- Greedy Choice Function

Optimal Substructure:

- If A is an optimal solution to a problem, then the components of A are optimal solutions to subproblems

Greedy Choice Function

- The rule for how to choose an item guaranteed be in the optimal solution

Greedy Algorithm Procedure:

- Apply the Greedy Choice Function to pick an item
- Identify your subproblem, then solve it

Minimum Spanning Trees

Readings: CLRS 21
(but not 21.1)

Prim's Algorithm

1. Start with an empty tree T and pick a start node and add it to T
2. Repeat $|V| - 1$ times:
   - Add the min-weight edge which connects a node in T with a node not in T

Implementation:

- Maintain nodes **not in** T in a min-heap (priority queue)
- Find the next closest node v (lowest edge weight) by extracting min from priority queue
- Each time node v (and edge) is added to the tree, update keys for neighbors still in min-heap
- Repeat until no nodes left in min-heap

Prim's Algorithm Implementation

1. Start with an empty tree T and pick a start node and add it to T
2. Repeat $|V| - 1$ times:
   - Add the min-weight edge which connects a node in T with a node not in T

Implementation:

initialize $d_v = \infty$ for each node v
add all nodes $v \in V$ to the priority queue PQ, using d_v as the key
pick a starting node s and set $d_s = 0$
while PQ is not empty:
    $v = \text{PQ.extractMin}()$
    for each $u \in V$ such that $(v, u) \in E$:
        if $u \in \text{PQ}$ and $w(v, u) < d_u$:
            $\text{PQ.decreaseKey}(u, w(v, u))$
            $u.\text{parent} = v$

each node also maintains a parent, initially NULL

key: minimum cost to connect u to nodes in PQ

Prim's Algorithm

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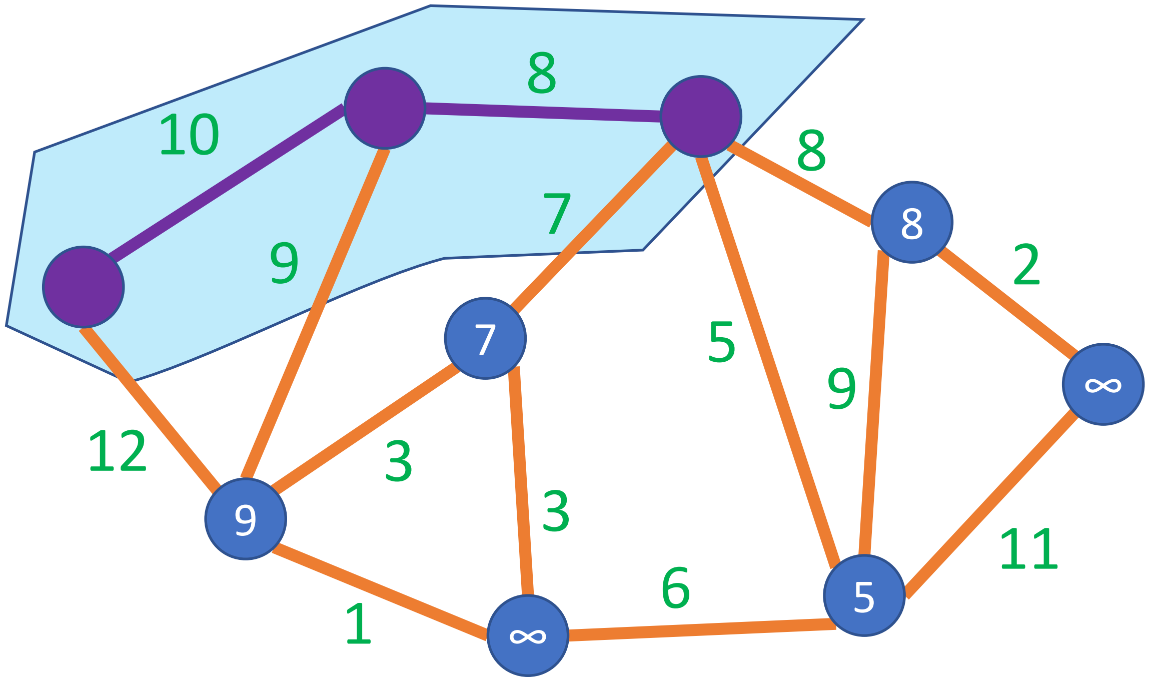

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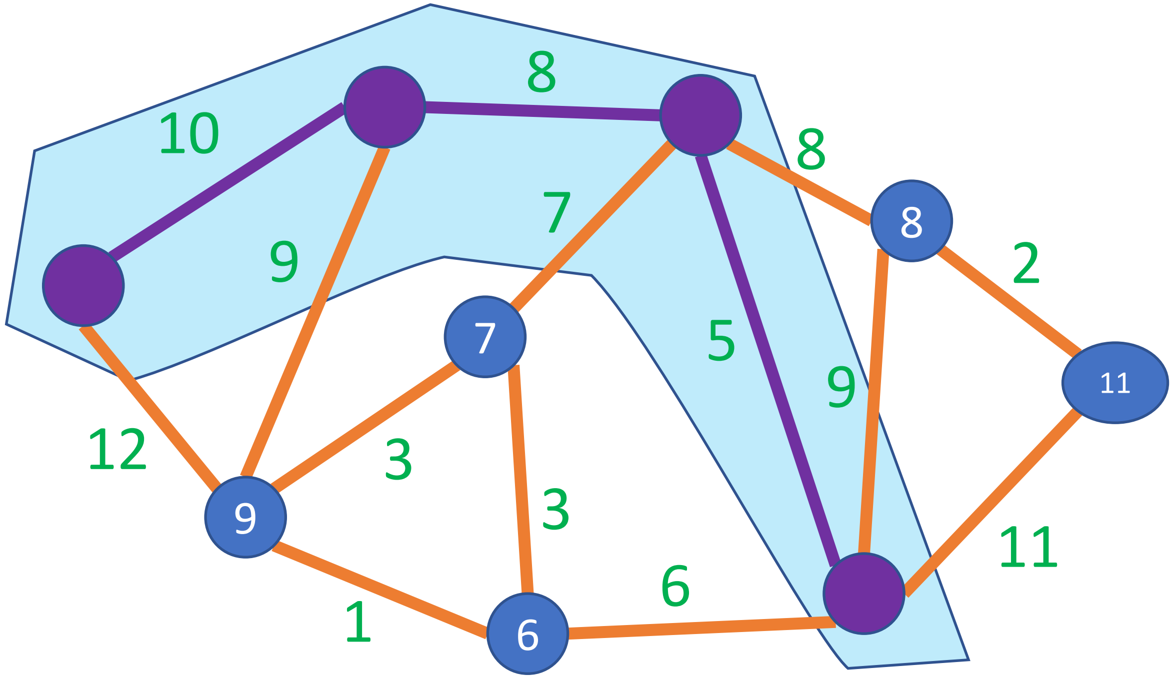

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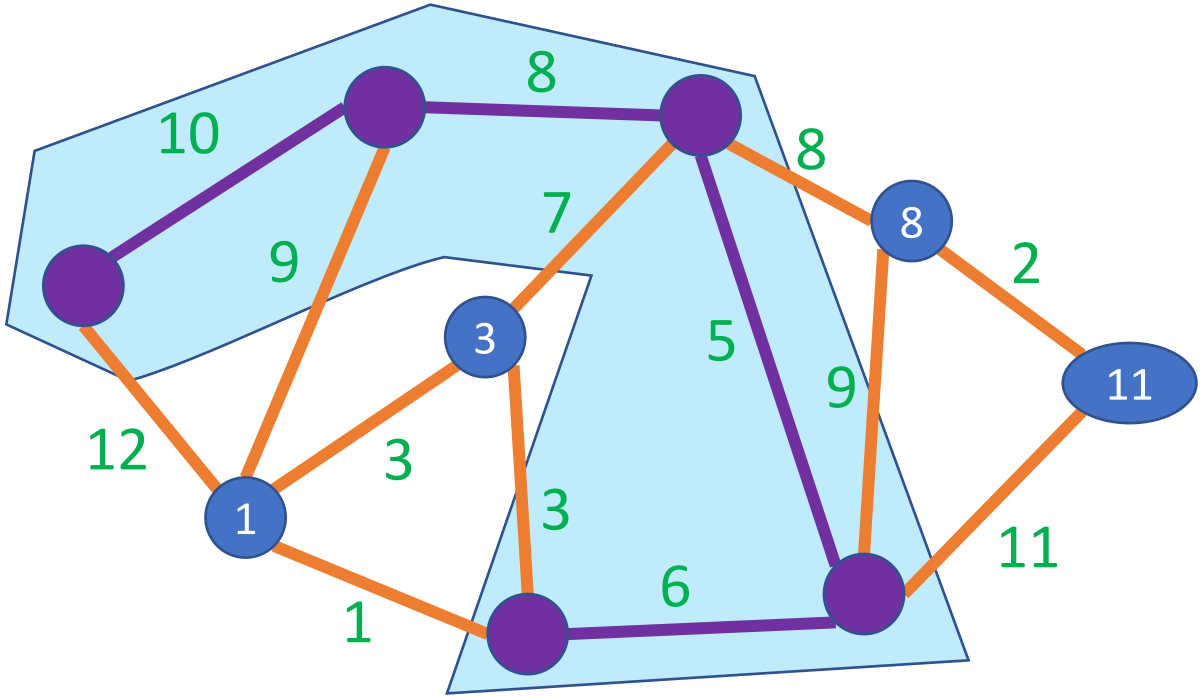

Reminder: Dijkstra's Algorithm Implementation

1. Start with an empty tree T and add the source to T
2. Repeat $|V| - 1$ times:
   - Add the "nearest" node not yet in T to T

Implementation:

initialize $d_v = \infty$ for each node v
add all nodes $v \in V$ to the priority queue PQ, using d_v as the key
set $d_s = 0$
while PQ is not empty:
  $v = \mathrm{PQ}.\mathrm{extractMin}()$
  for each $u \in V$ such that $(v, u) \in E$:
    if $u \in \mathrm{PQ}$ and $d_v + w(v, u) < d_u$:
      $\mathrm{PQ}.\mathrm{decreaseKey}\left(u, d_v + w(v, u)\right)$
      $u.\mathrm{parent} = v$

each node also maintains a parent, initially `NULL`

key: length of shortest path $s \rightarrow u$ using nodes in PQ

Prim's Algorithm Implementation

1. Start with an empty tree T and pick a start node and add it to T
2. Repeat $|V| - 1$ times:
   - Add the min-weight edge which connects a node in T with a node not in T

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            $u.\mathrm{parent} = v$

each node also maintains a parent, initially `NULL`

key: minimum cost to connect u to nodes in PQ

Prim's Algorithm Running Time

Same as for Dijkstra's Shortest Path algorithm!

Implementation (with nodes in the priority queue):

initialize $d_v = \infty$ for each node v
add all nodes $v \in V$ to the priority queue PQ, using d_v as the key
pick a starting node s and set $d_s = 0$
while PQ is not empty:
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Initialization: $O(|V|)$

$|V|$ iterations

$O(\log|V|)$

$|E|$ iterations <u>total</u>

$O(\log|V|)$

Using indirect heaps

Overall running time: $O(|V| \log|V| + |E| \log|V|) = O(|E| \log|V|)$

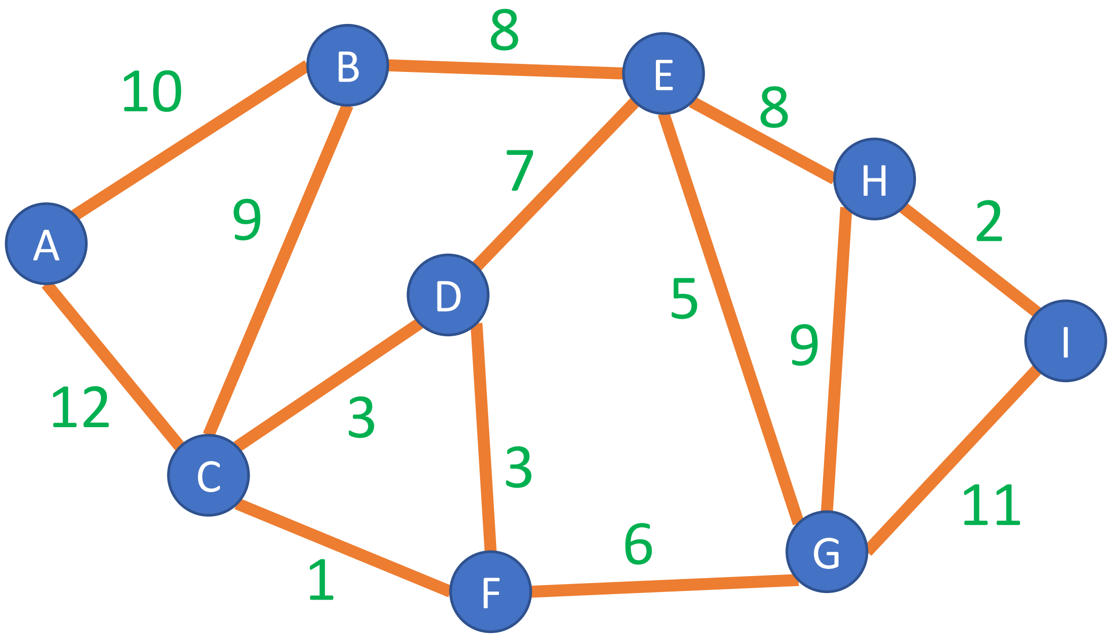
Kruskal’s MST Algorithm

Readings: CLRS first part of 21.2

Kruskal's Algorithm

The *Greedy Choice* for Kruskal's

1. Start with an empty set of edges T
2. Repeatedly add to T the lowest-weight edge that does not create a cycle. (Stop when we've added $n-1$ edges.)



Kruskal's Algorithm

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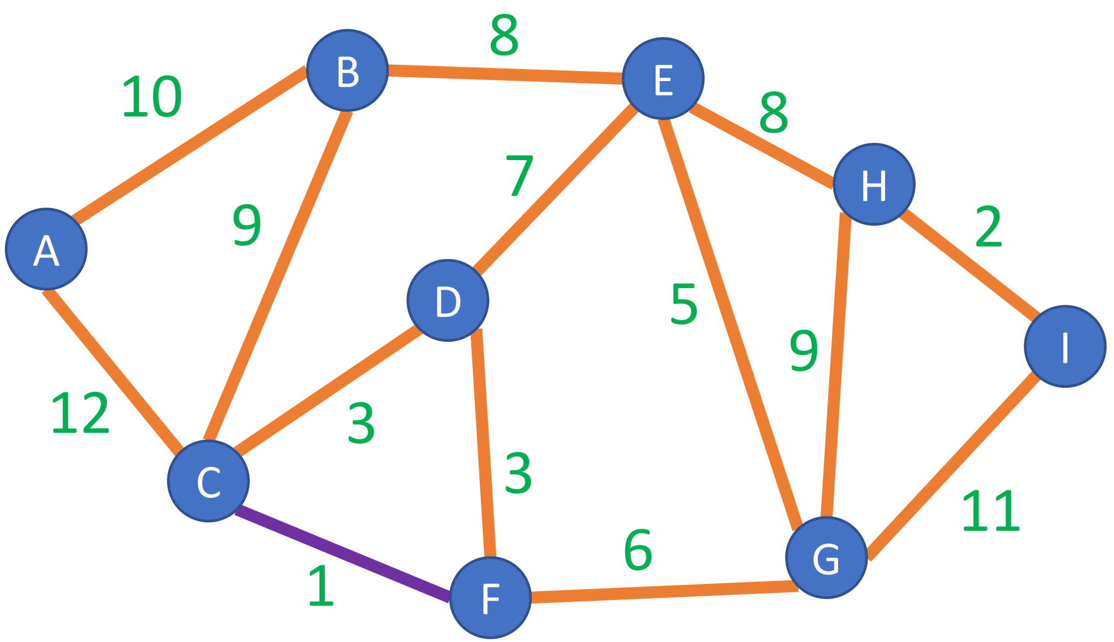

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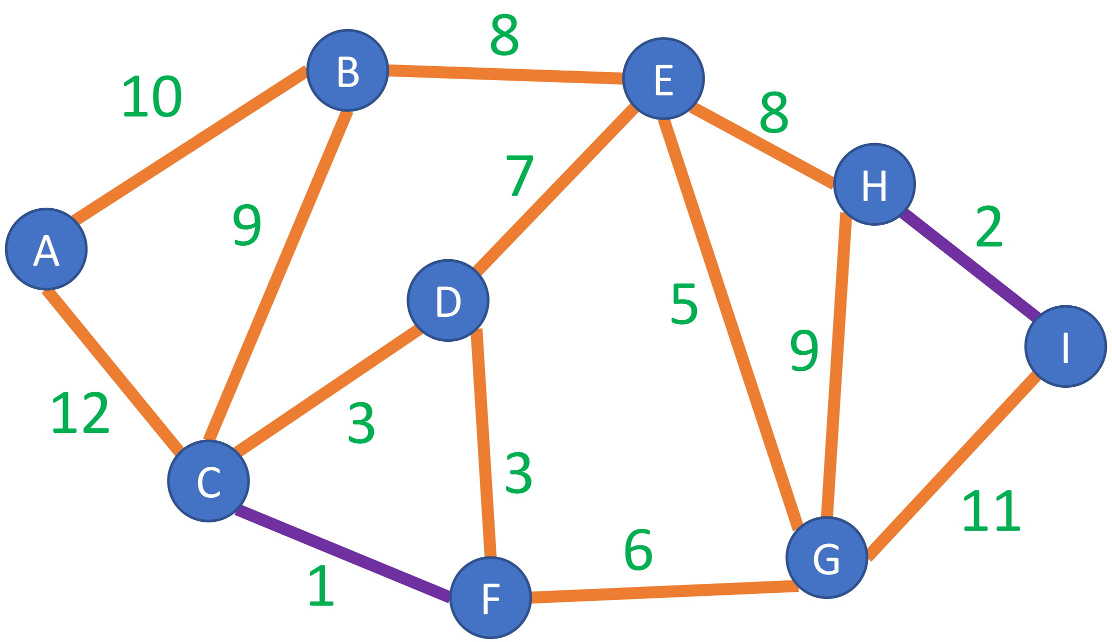

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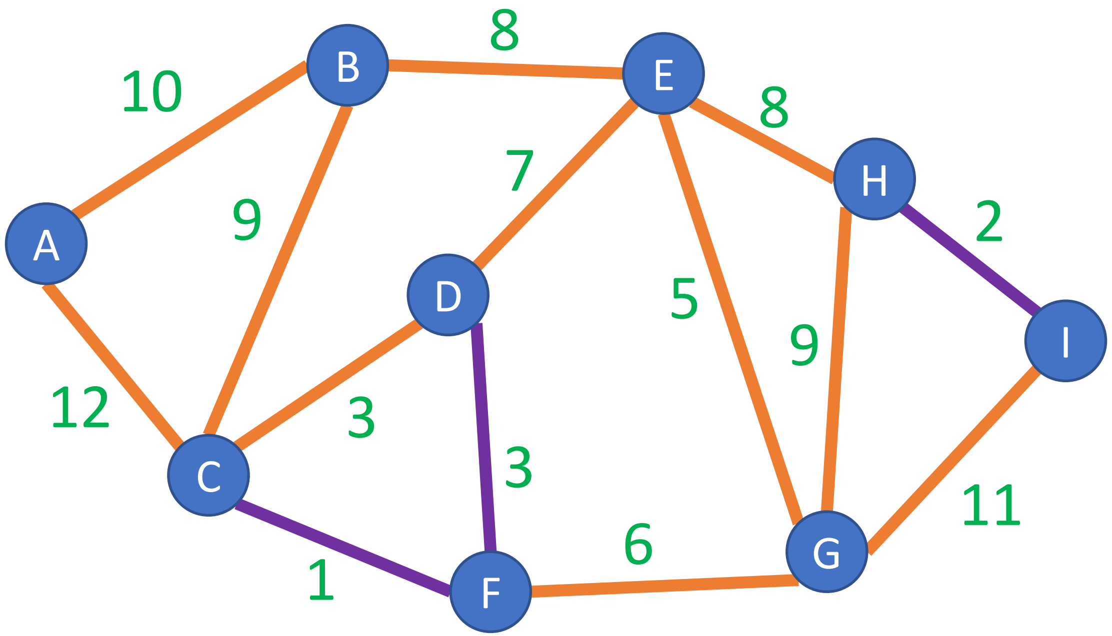

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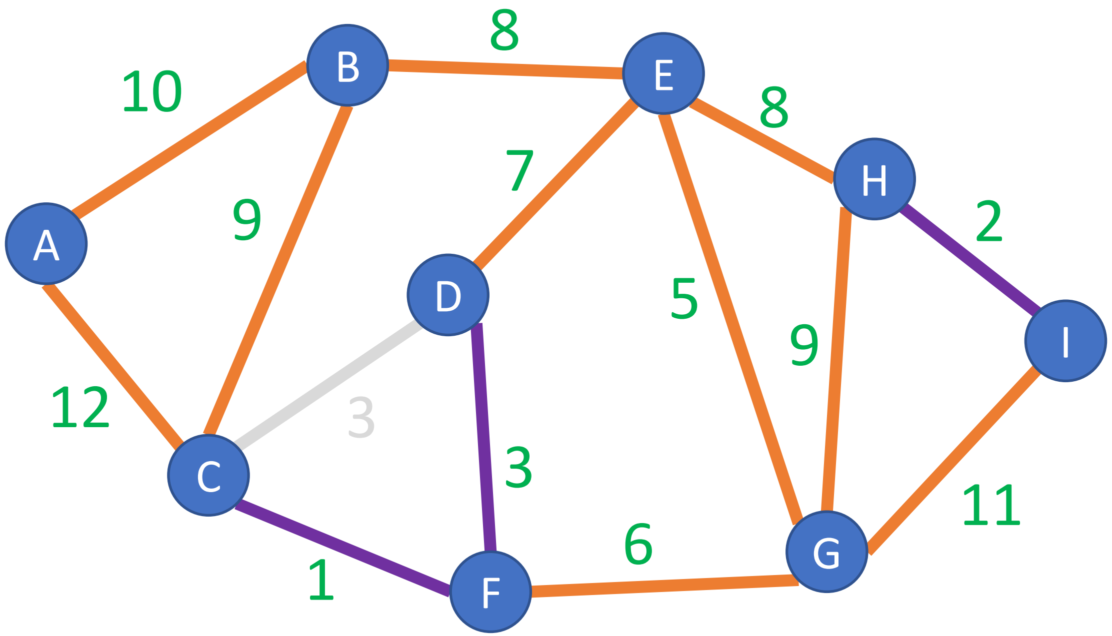

Edge forms a cycle, so do not include

Kruskal's Algorithm

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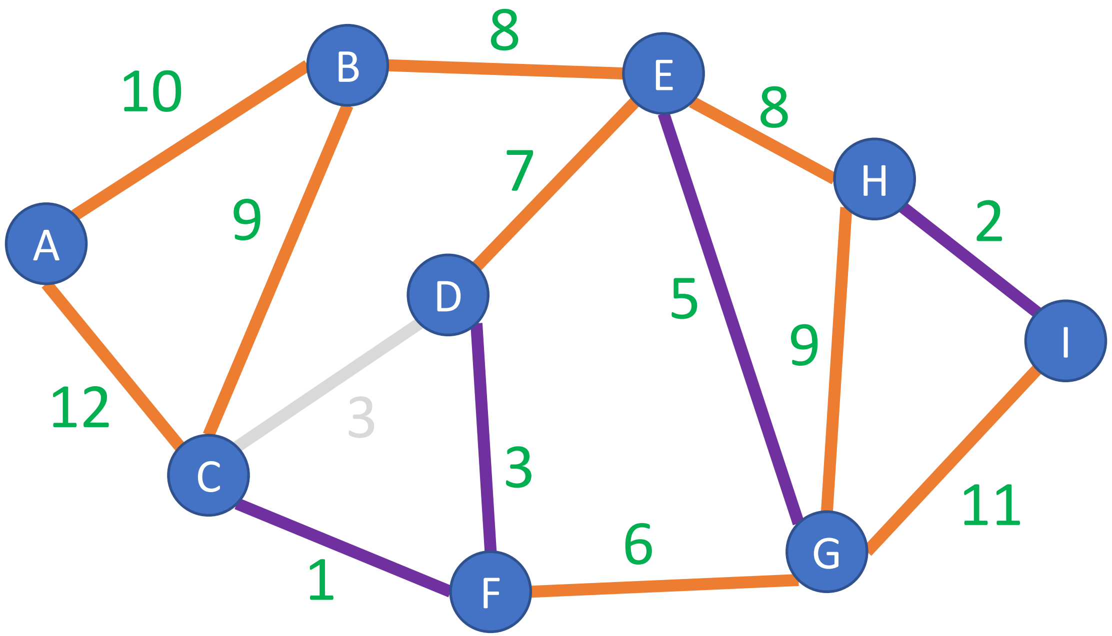

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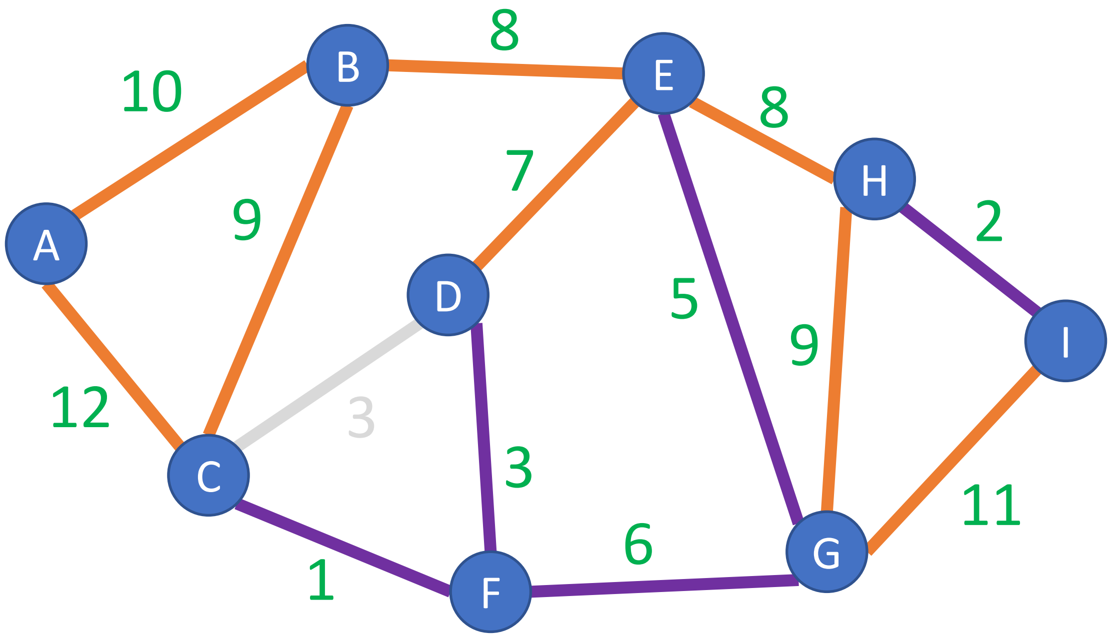

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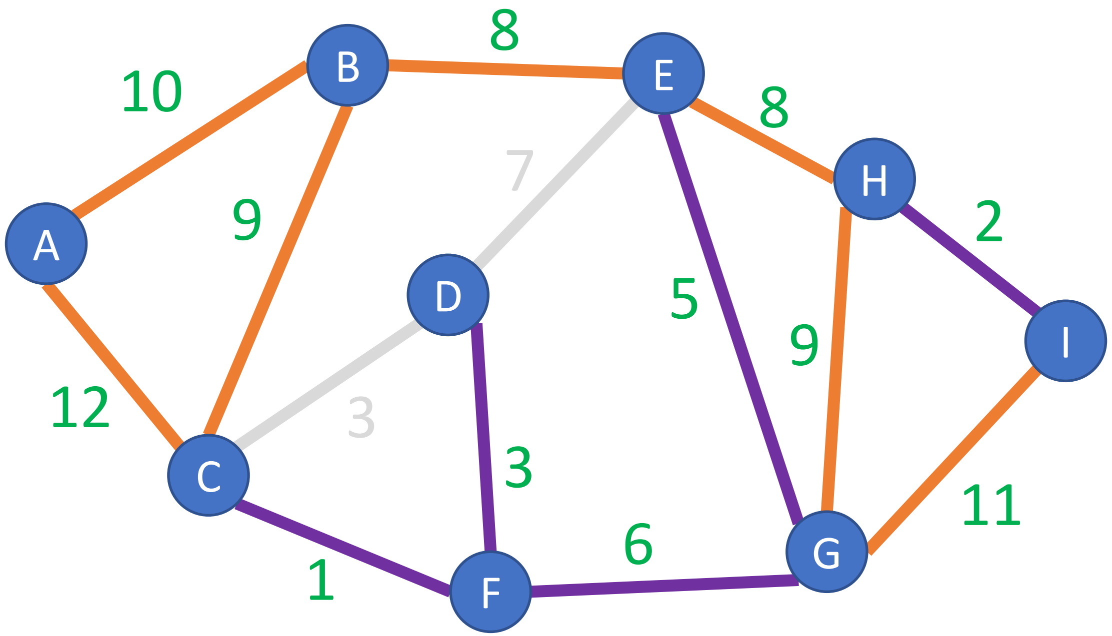

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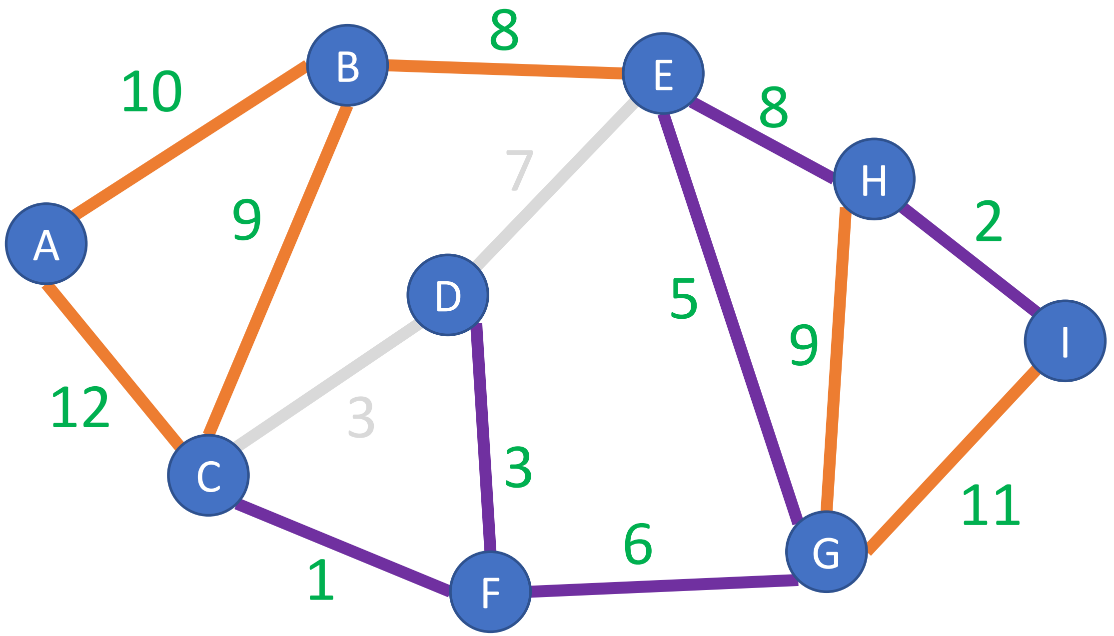

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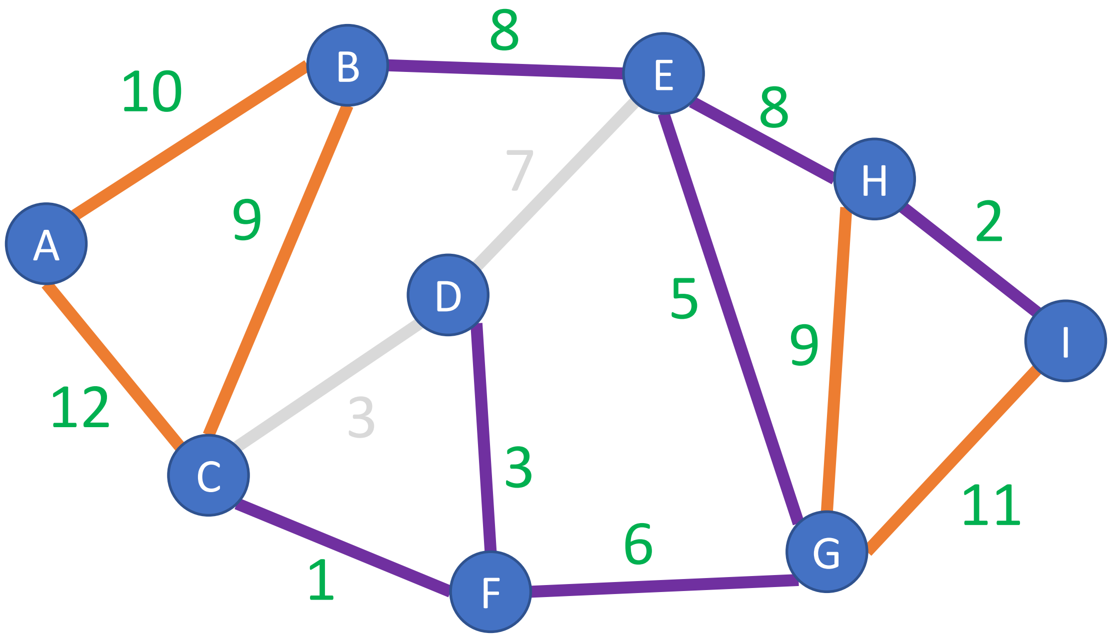

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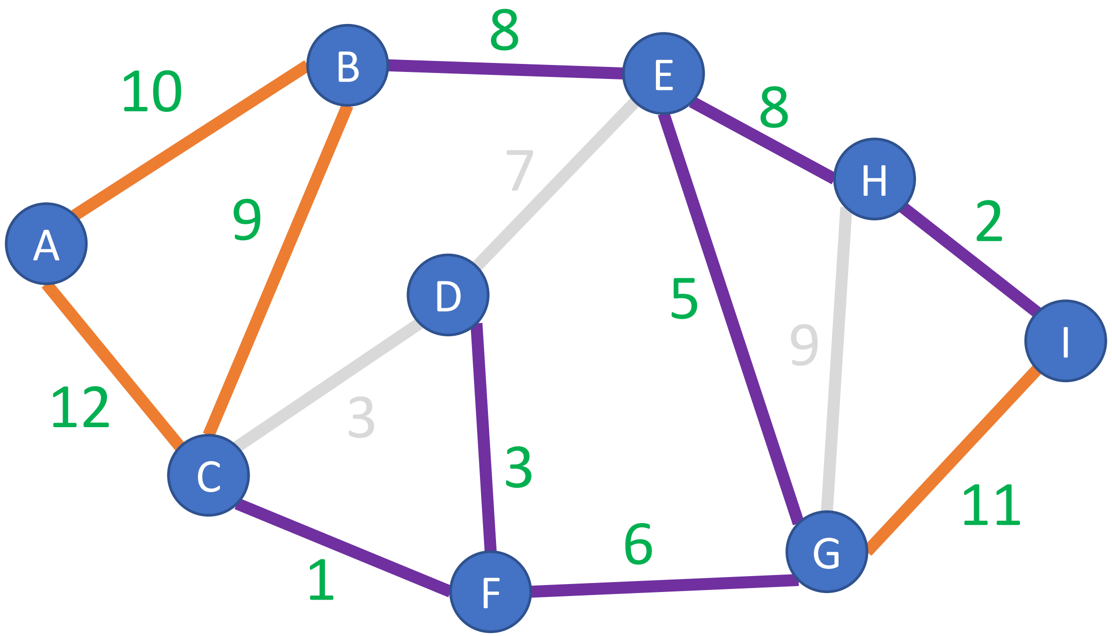

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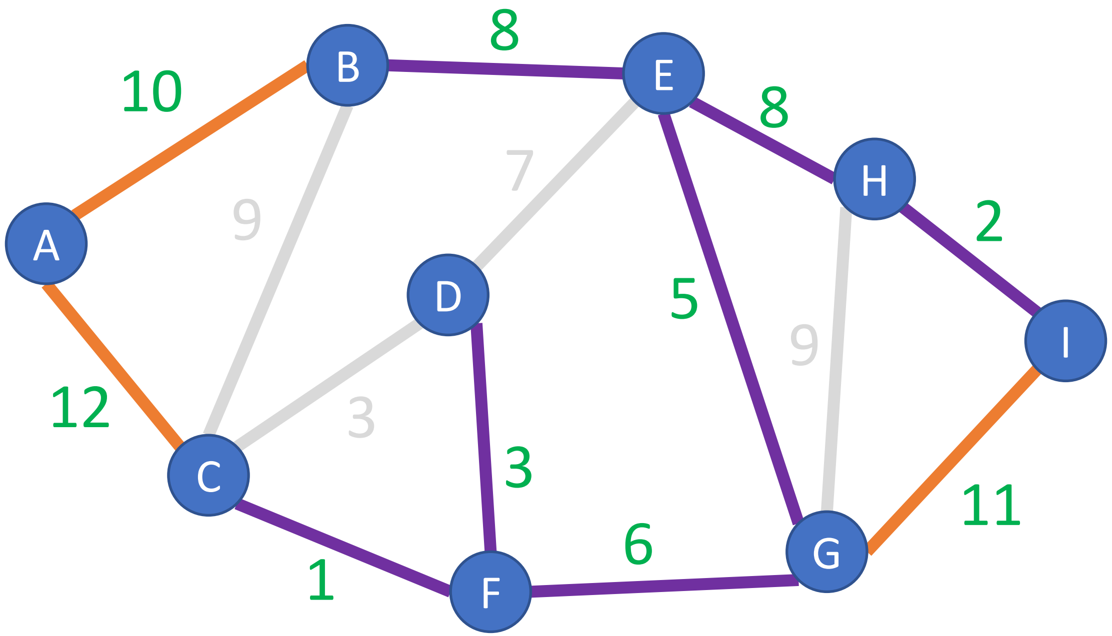

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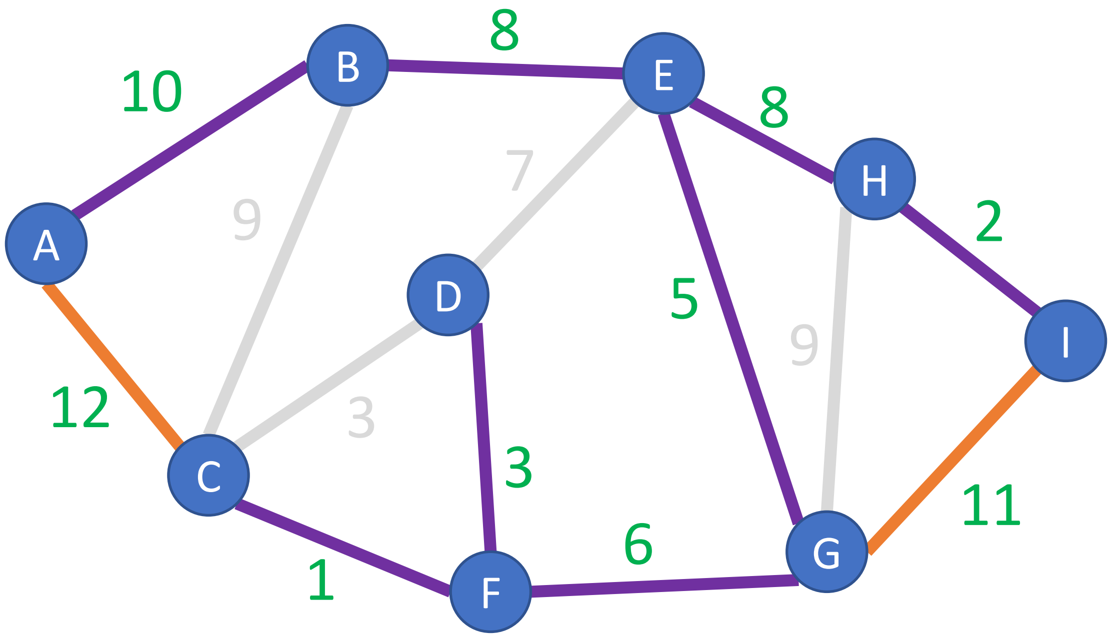


Now $n - 1$ edges have been added.
All nodes are connected.
Algorithm is done!

Kruskal's Algorithm

1. Start with an empty tree T
2. Repeatedly add to T the lowest-weight edge that does not create a cycle

Implementation: iterate over each of the edges in the graph (sorted by weight), and maintain nodes in a union-find (also called disjoint-set) data structure:

- Data structure that tracks elements partitioned into different sets
- **Union:** Merges two sets into one
- **Find:** Given an element, return the index of the set it belongs to
- Both "union" and "find" operations are very fast

Time complexity: $O(\alpha(n))$,
where α is the "inverse Ackermann function" (extremely slow-growing function)
for all "practical" n, $\alpha(n) < 5$ (e.g., for all $n < 2^{2^{2^{65536}}} - 3$)

Union/Find and Disjoint Sets

An Abstract Data Type (ADT) for a collection of sets of any kind of item, where an item can only belong to one of the sets

- We'll assume each item is identified by a unique integer value

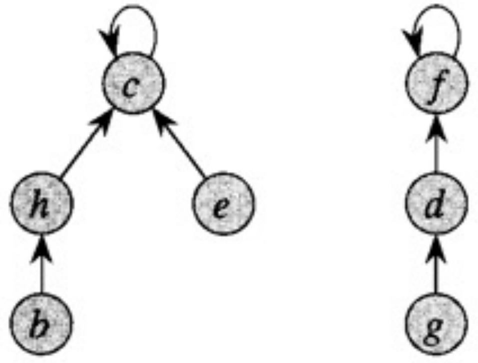
Need to support the following operations

- void makeSet(int n) // construct n independent sets
- int findSet(int i) // given i, which set does i belong to?
- void union(int i, int j) // merge sets containing i and j

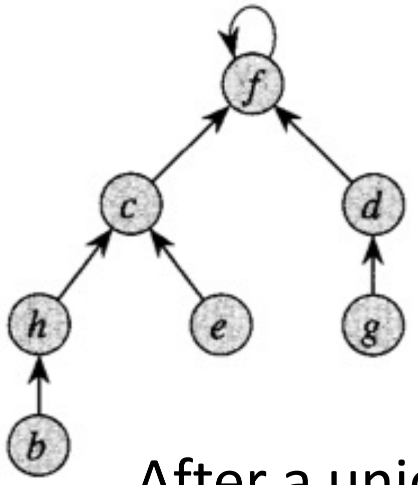
Union/Find and Disjoint Sets

Represent Sets As Trees

- Represent each set as a tree
- Identify set by its root node's ID (its "label")
  - findSet() means tracing up to root
  - union() makes one root child of the other root



Two sets



After a union

Time Complexity: Kruskal's Algorithm

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- **Overall running time:** $O(|E| \log |E|) = O(|E| \log |V|)$

$$|E| \leq |V|^2 \Rightarrow \log|E| = O(\log|V|)$$

More on Implementation for Kruskal's

Let *EL* be the set of edges sorted ascending by weight

Consider each vertex to be in a tree of size 1

For each edge *e* in *EL*

    T1 = tree ID for vertex *head(e)*

    T2 = tree ID for vertex *tail(e)*

    if (*T1* != *T2*)   *// the nodes are not in the same Tree*

        Add *e* to the output set of edges *T* (which becomes the MST)

        Combine trees *T1* and *T2*

Seems simple, no?

- But, how do you keep track of what tree a vertex is in?
- Trees are sets of vertices. Need to findset(v) and "union" two sets

Proof of Correctness: Exchange Argument

Common technique to show correctness of a greedy algorithm

General idea: argue that at every step, the greedy choice is part of some optimal solution

Approach: Start with an arbitrary optimal solution and show that exchanging an item from the optimal solution with your greedy choice makes the new solution no worse (i.e., the greedy choice is as good as the optimal choice)

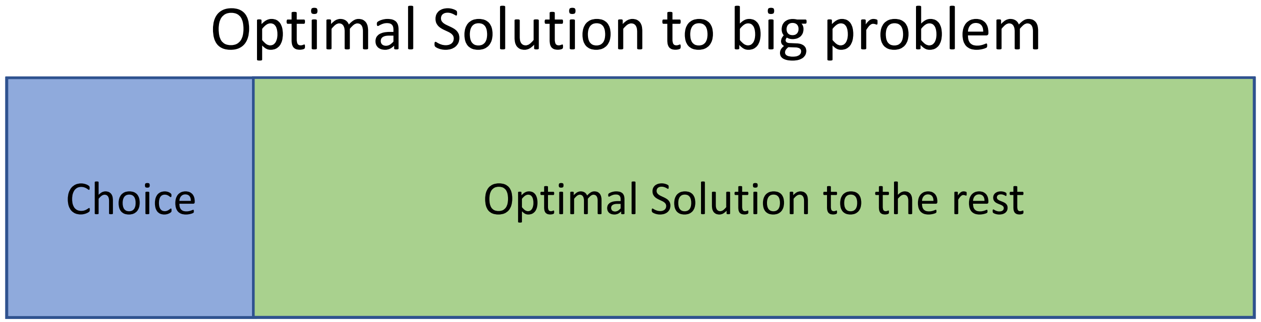
Exchange argument

Shows correctness of a greedy algorithm

Idea:

- Show exchanging an item from an arbitrary optimal solution with your greedy choice makes the new solution no worse
- How to show my sandwich is at least as good as yours:
  - Show: “I can remove any item from your sandwich, and it would be no worse by replacing it with the same item from my sandwich”

Greedy Algorithms



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Greedy Choice Function

- The rule for how to choose an item guaranteed be in the optimal solution

Greedy Algorithm Procedure:

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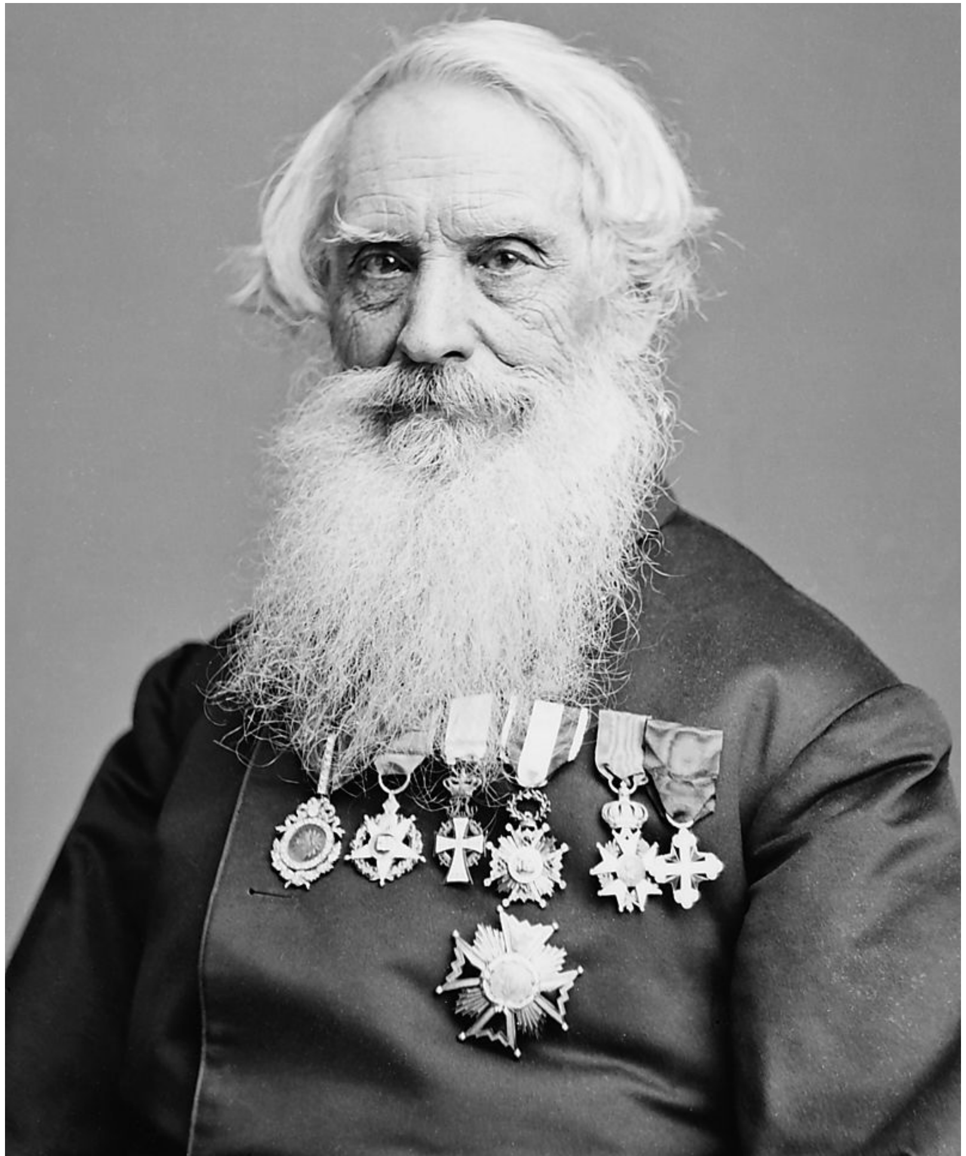
- Identify your subproblem, then solve it

Sam Morse

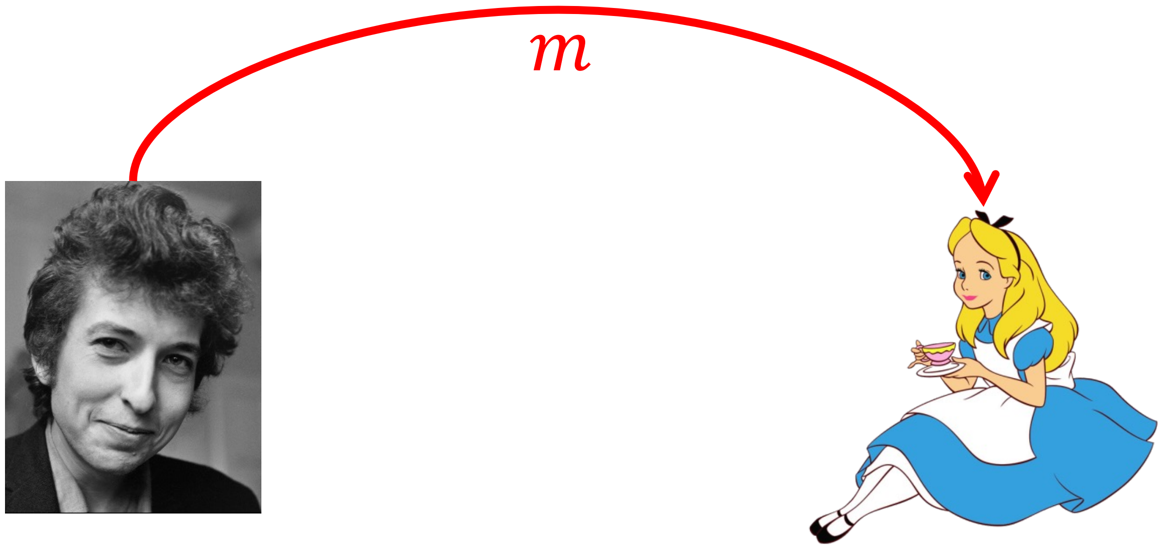
Engineer
and artist



Message Encoding

Problem: need to electronically send a message to two people at a distance.

Channel for message is binary (either on or off)



How can we do it?

wiggle, wiggle, wiggle like a gypsy queen
wiggle, wiggle, wiggle all dressed in green

Take the message, send it over character-by-character with an encoding

Character Frequency	Encoding
a: 2	0000
d: 2	0001
e: 13	0010
g: 14	0011
i: 8	0100
k: 1	0101
l: 9	0110
n: 3	0111
p: 1	1000
q: 1	1001
r: 2	1010
s: 3	1011
u: 1	1100
w: 6	1101
y: 2	1110

How efficient is this?

wiggle wiggle wiggle like a gypsy queen
wiggle wiggle wiggle all dressed in green

Each character requires 4 bits

$$\ell_c = 4$$

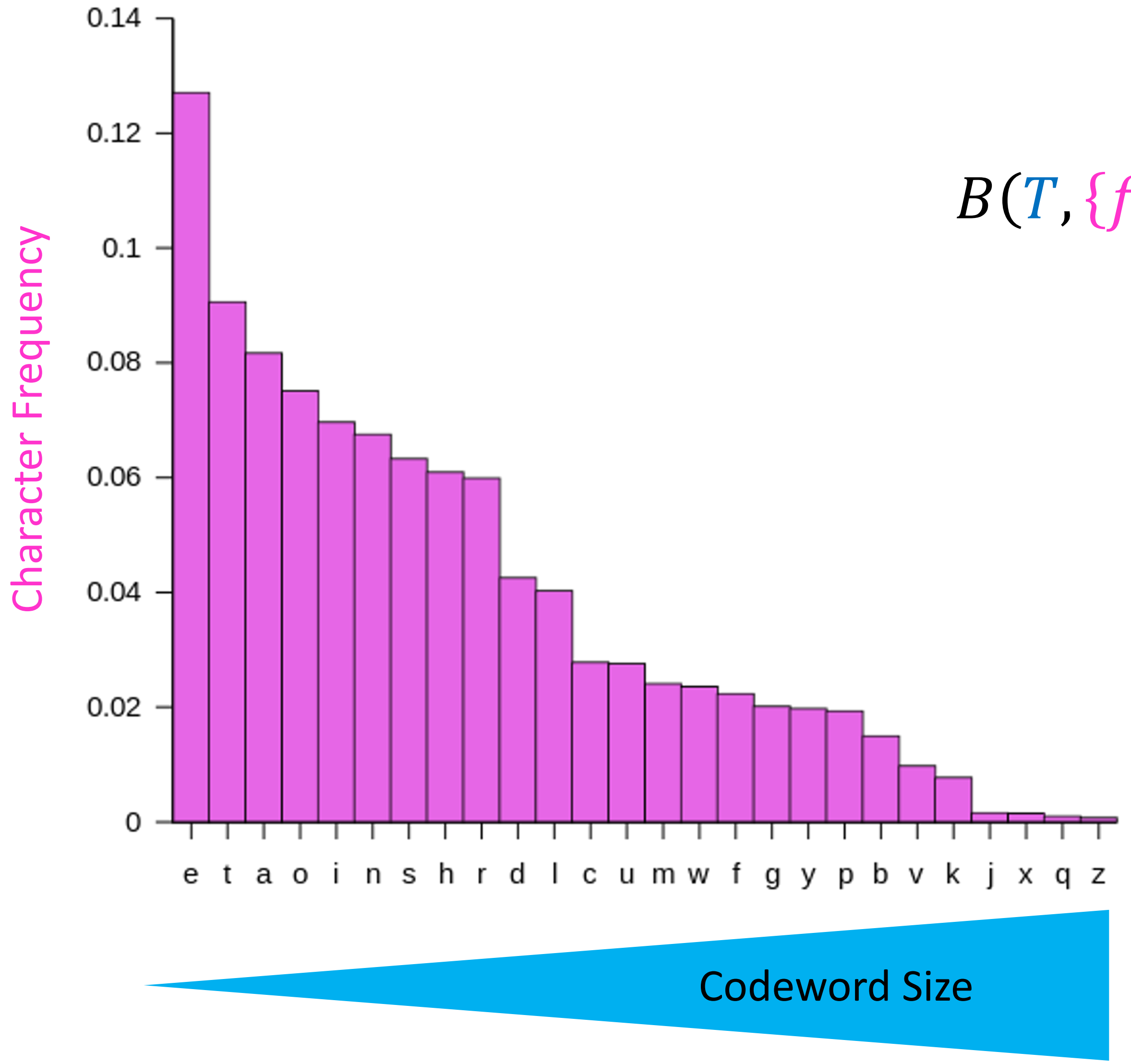
Cost of encoding:

$$B(T, \{f_c\}) = \sum_{character\ c} \ell_c f_c = 68 \cdot 4 = 272$$

Better Solution: Allow for different characters to have different-size encodings (high frequency → short code)

Character Frequency	Encoding
a: 2	0000
d: 2	0001
e: 13	0010
g: 14	0011
i: 8	0100
k: 1	0101
l: 9	0110
n: 3	0111
p: 1	1000
q: 1	1001
r: 2	1010
s: 3	1011
u: 1	1100
w: 6	1101
y: 2	1110

More efficient coding

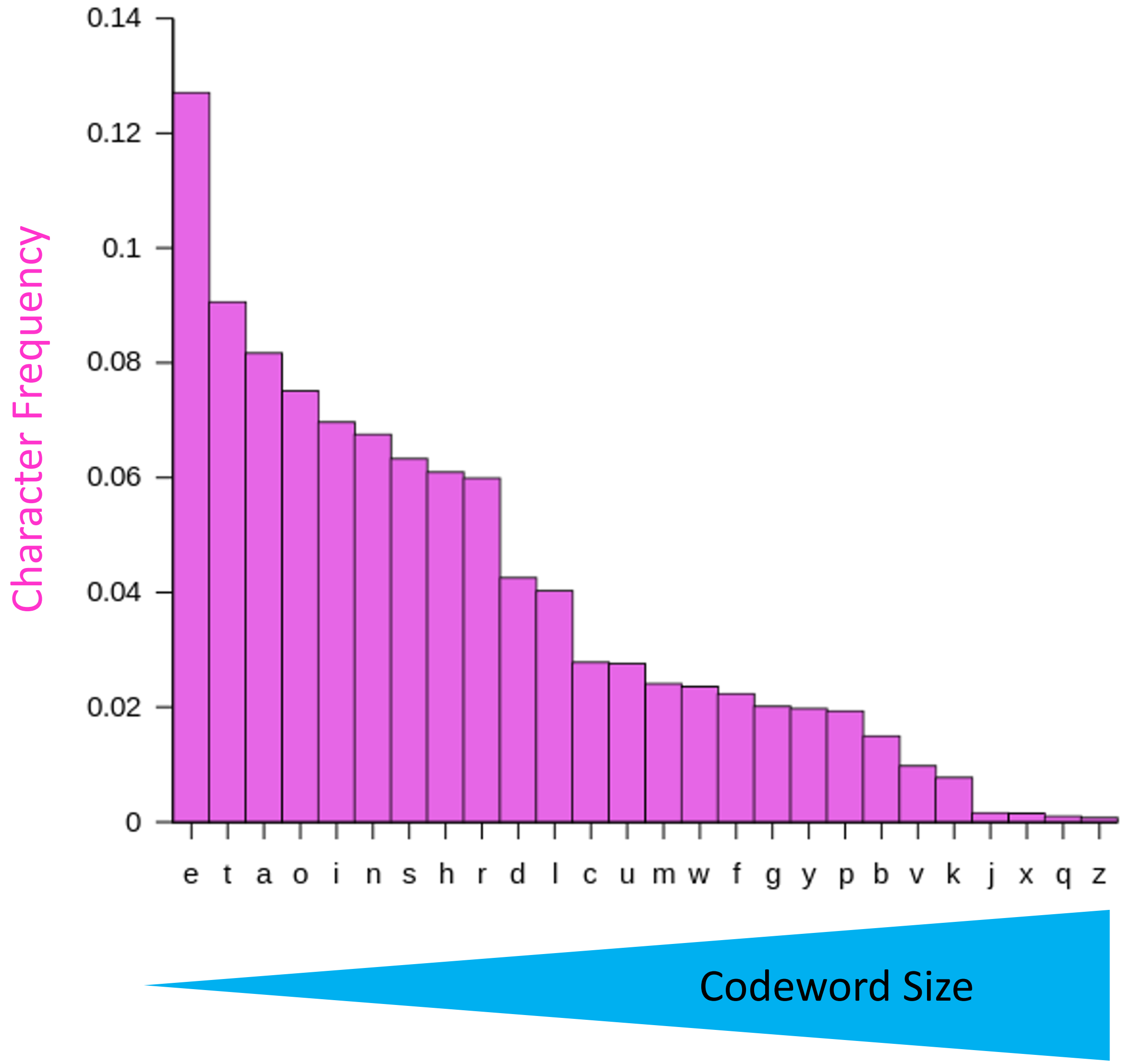


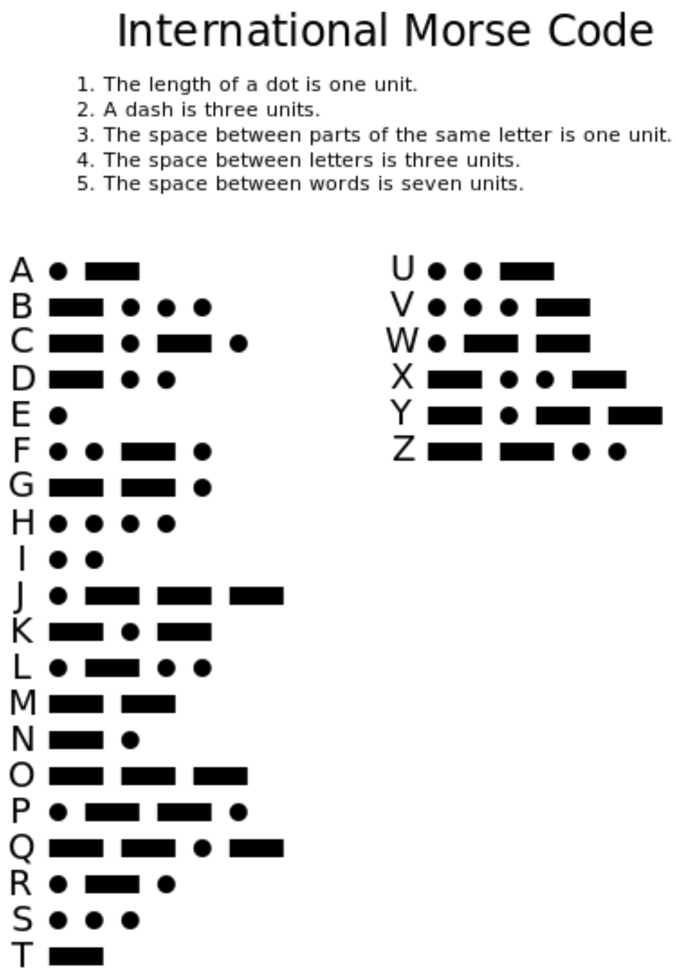
$$B(T, \{f_c\}) = \sum_{\text{character } c} \ell_c f_c$$

When this is big

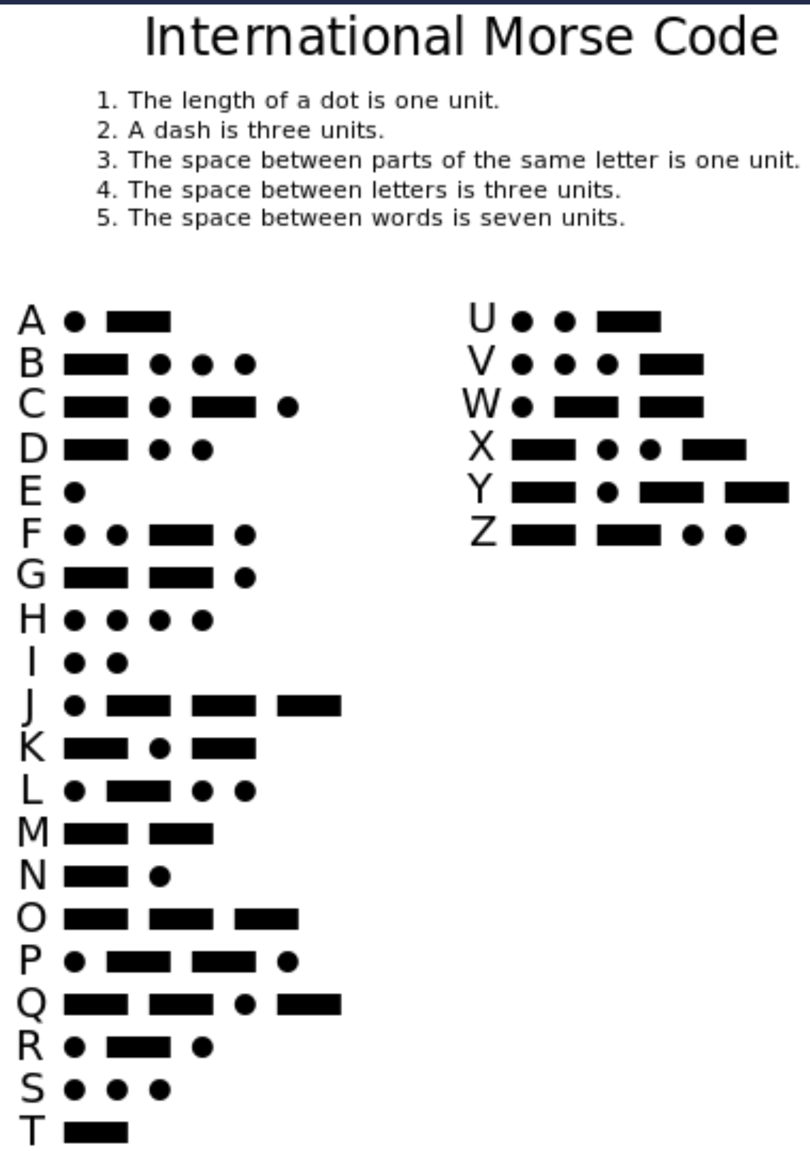
Make this small

Morse Code

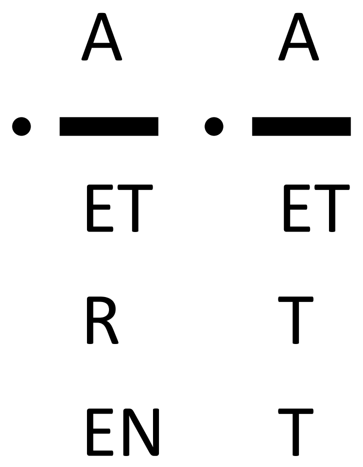

Character Frequency
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
e t a o i n s h r d l c u m w f g y p b v k j x q z
Codeword Size


International Morse Code
1. The length of a dot is one unit.
2. A dash is three units.
3. The space between parts of the same letter is one unit.
4. The space between letters is three units.
5. The space between words is seven units.
A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
Q
R
S
T
U
V
W
X
Y
Z

Problem with Morse Code



Decode:



Ambiguous Decoding

Prefix-Free Code

A prefix-free code is codeword table T such that for any two characters c_1, c_2, if $c_1 \neq c_2$ then $code(c_1)$ is not a prefix of $code(c_2)$

g	0
e	10
l	110
i	1110
w	11110
…	…

11110111000011010

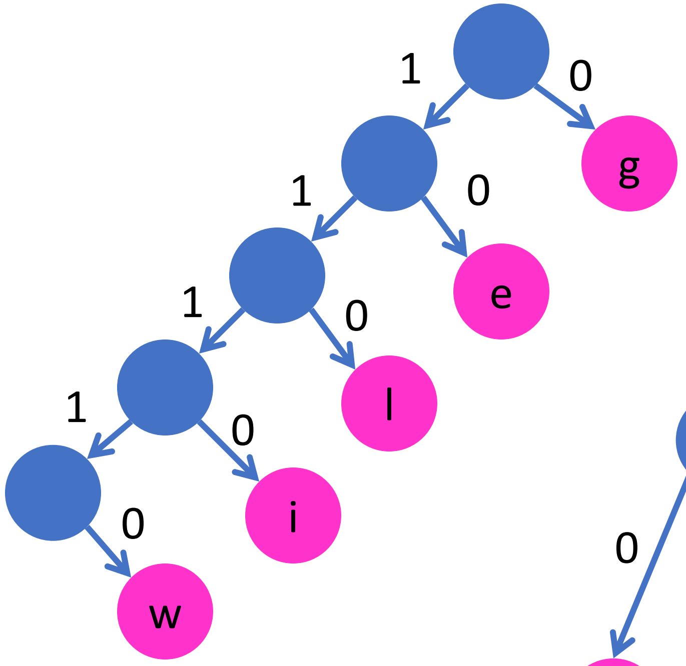
w i gg l e

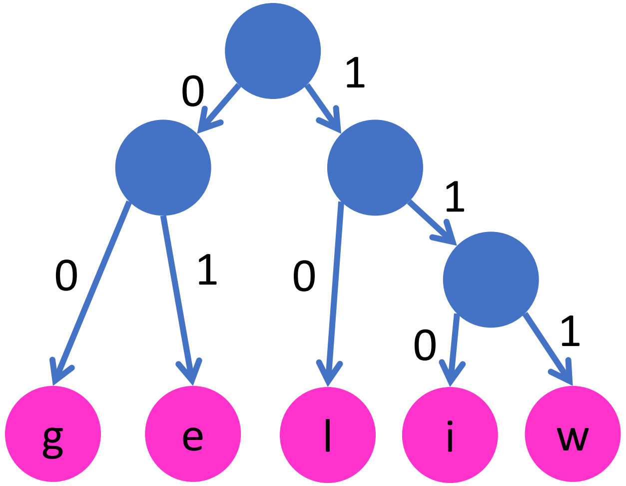
Binary Trees = Prefix-free Codes

I can represent any prefix-free code as a binary tree

I can create a prefix-free code from any binary tree

g	0
e	10
l	110
i	1110
w	11110
…	…





g	00
e	01
l	10
i	110
w	111
…	…

Goal: Shortest Prefix-Free Encoding

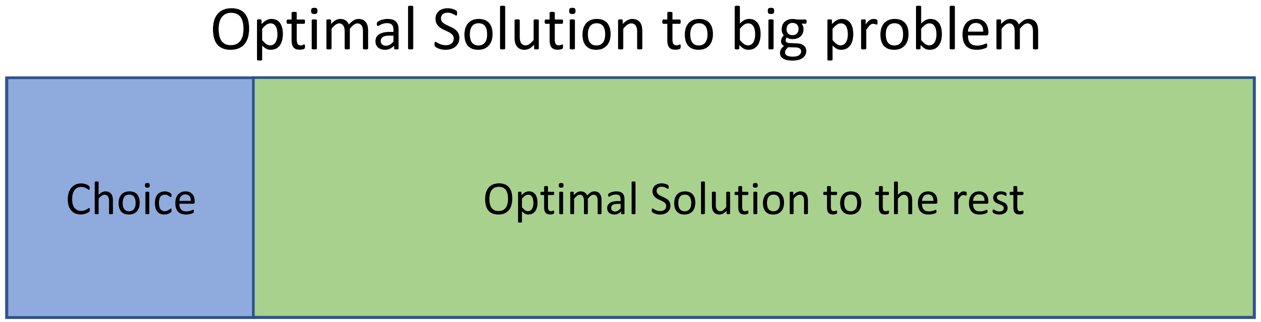
Input: A set of character frequencies $\{f_c\}$

Output: A prefix-free code T which minimizes

$$B(T, \{f_c\}) = \sum_{character\ c} \ell_c f_c$$

Huffman Coding!!

Greedy Algorithms



Require two things:

- Optimal Substructure
- Greedy Choice Function

Optimal Substructure:

- If A is an optimal solution to a problem, then the components of A are optimal solutions to subproblems

Greedy Choice Function

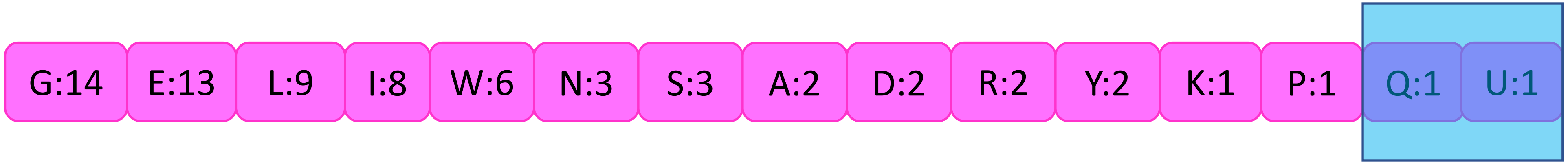
- The rule for how to choose an item guaranteed be in the optimal solution

Greedy Algorithm Procedure:

- Apply the Greedy Choice Function to pick an item
- Identify your subproblem, then solve it

Huffman Algorithm

Choose the least frequent pair, combine into a subtree



Huffman Algorithm

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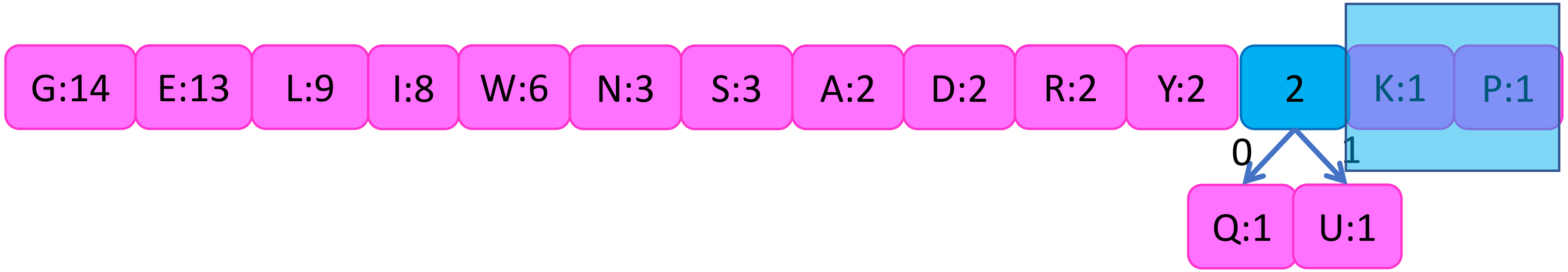
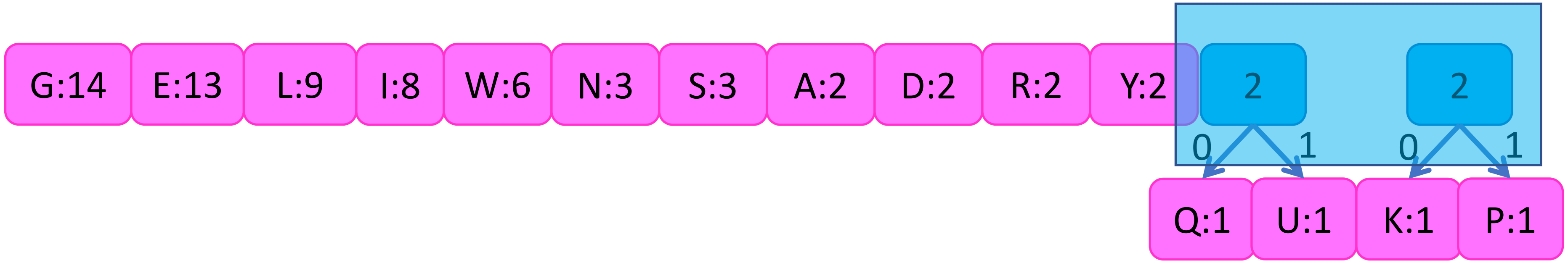


Subproblem of size $n - 1$!

Huffman Algorithm

Choose the least frequent pair, combine into a subtree



Huffman Algorithm

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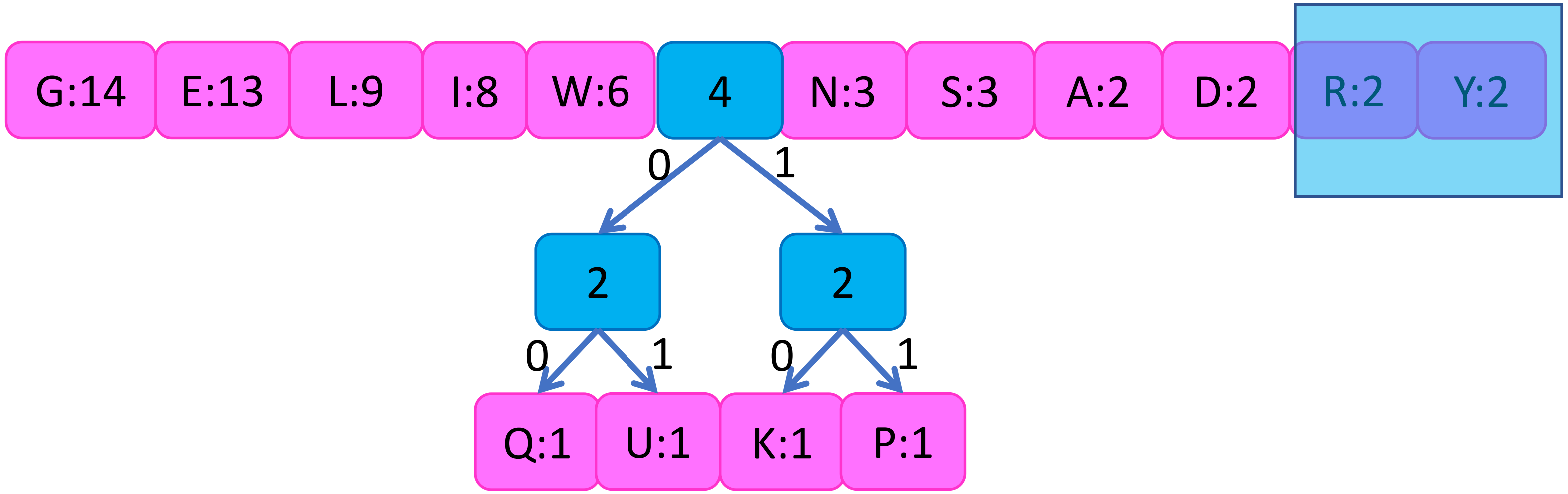

Huffman Algorithm

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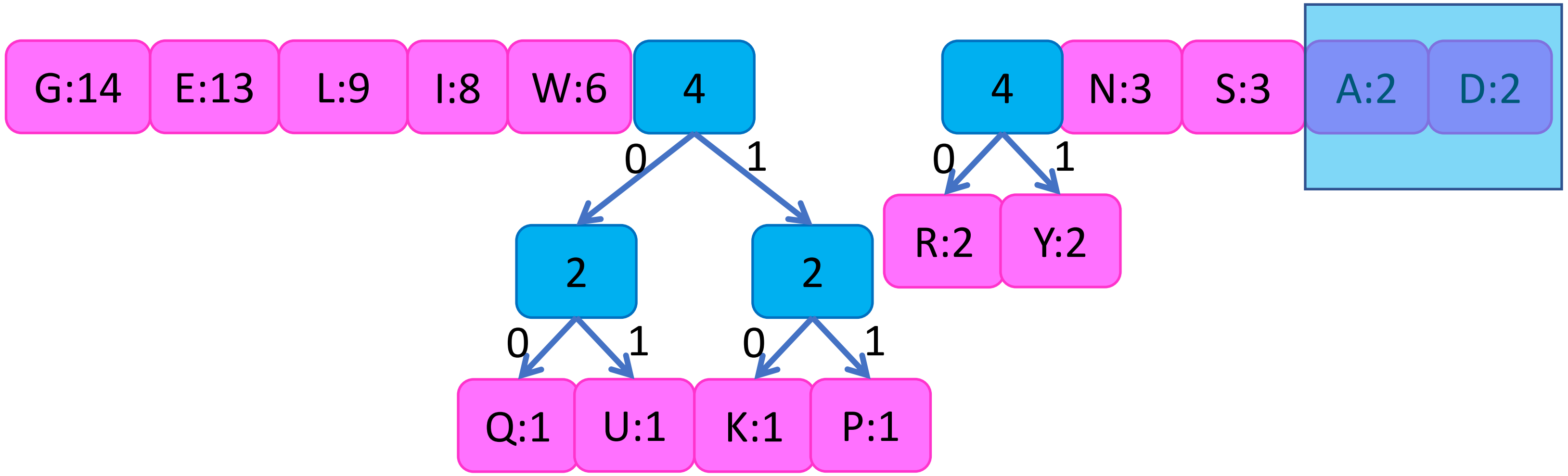

Huffman Algorithm

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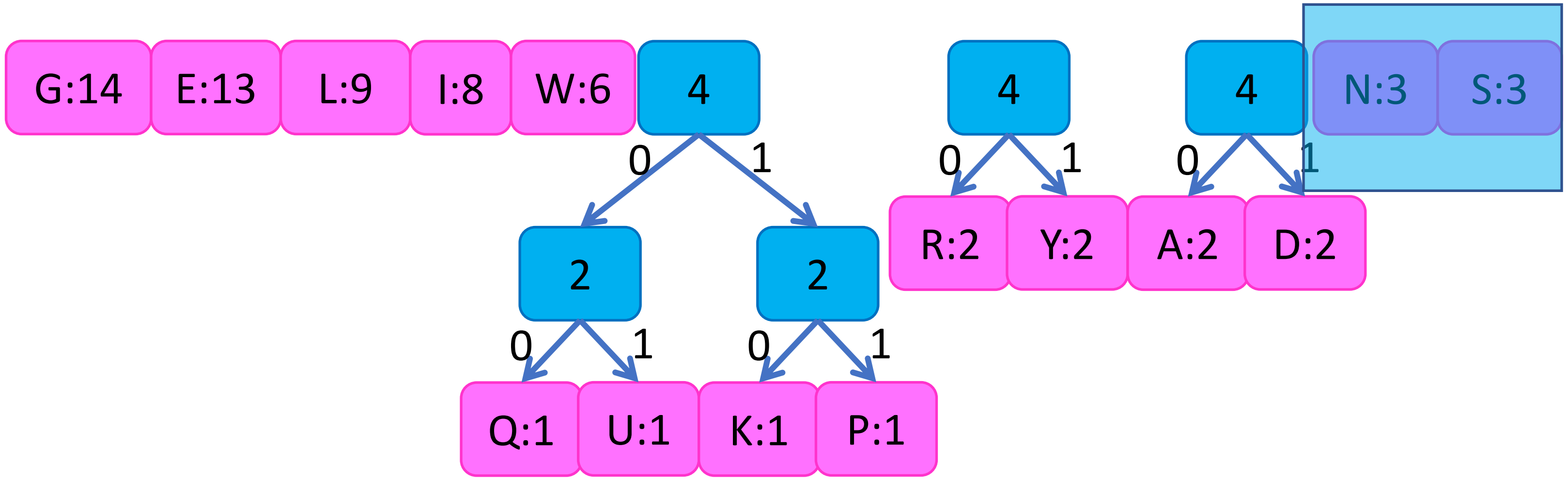
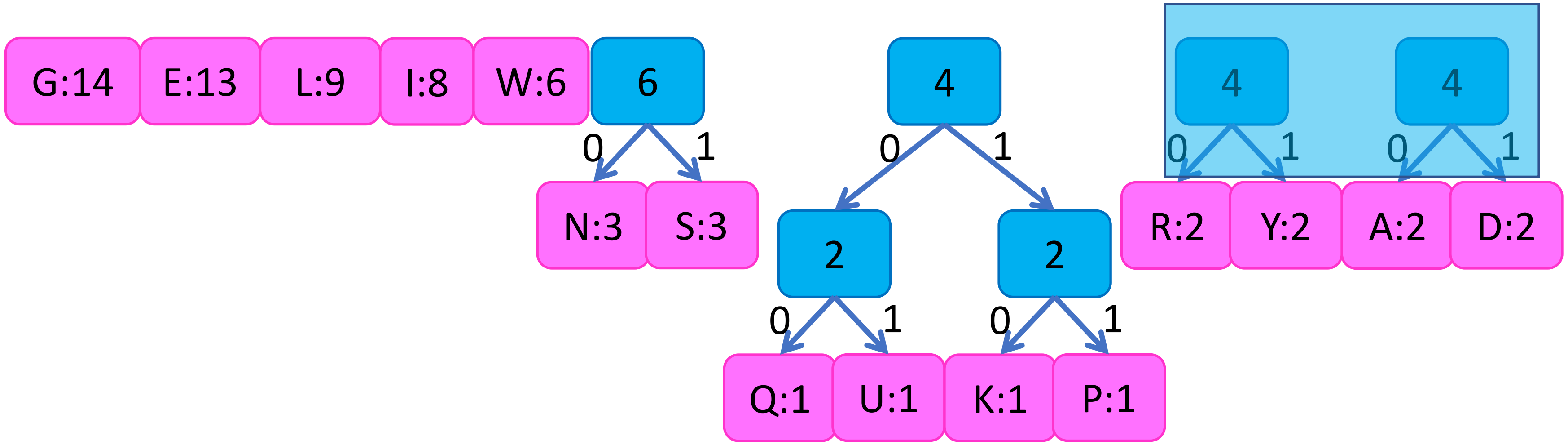


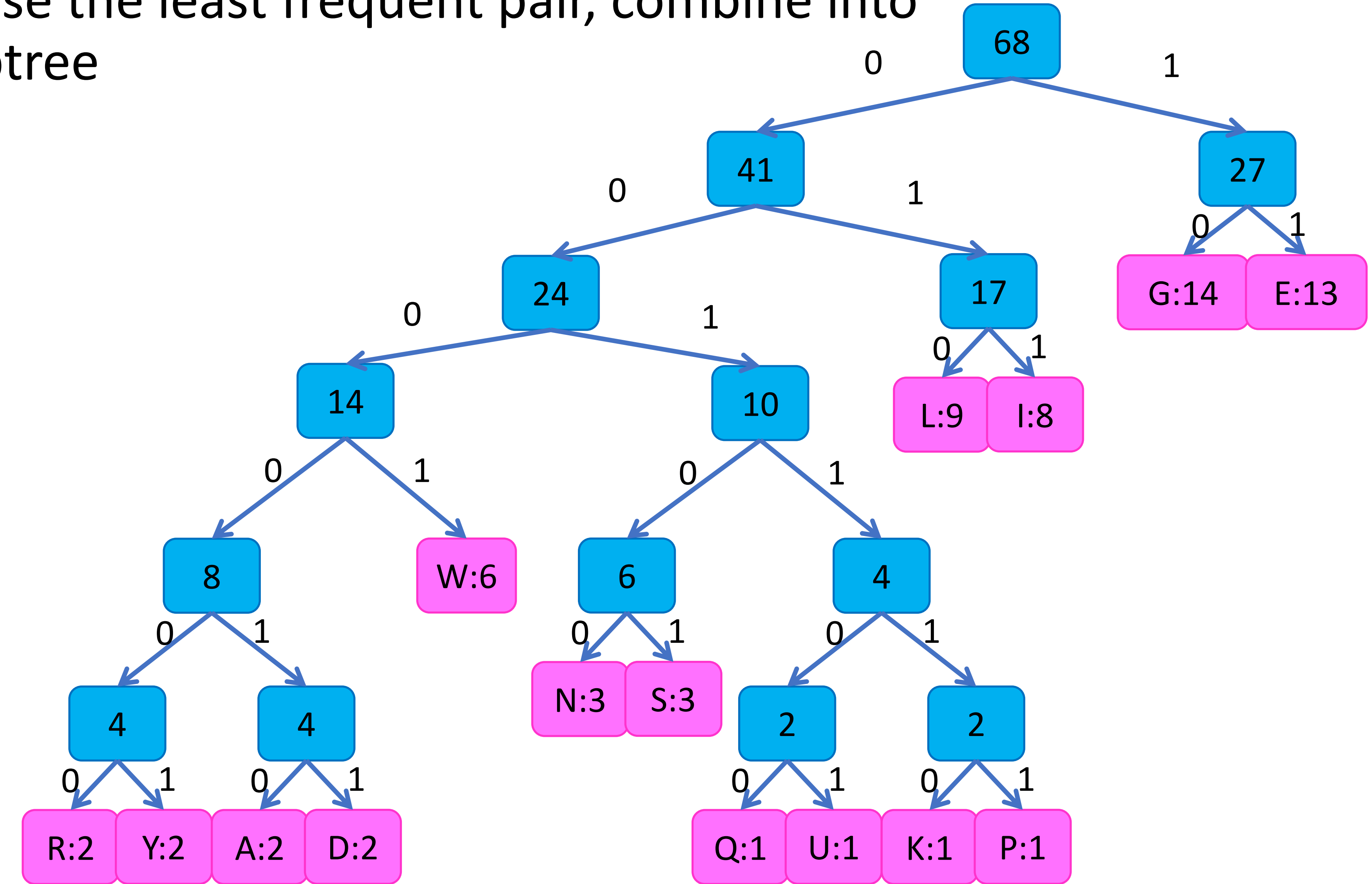
Huffman Algorithm

Choose the least frequent pair, combine into a subtree



Huffman Algorithm

Choose the least frequent pair, combine into a subtree



Exchange argument

Shows correctness of a greedy algorithm

Idea:

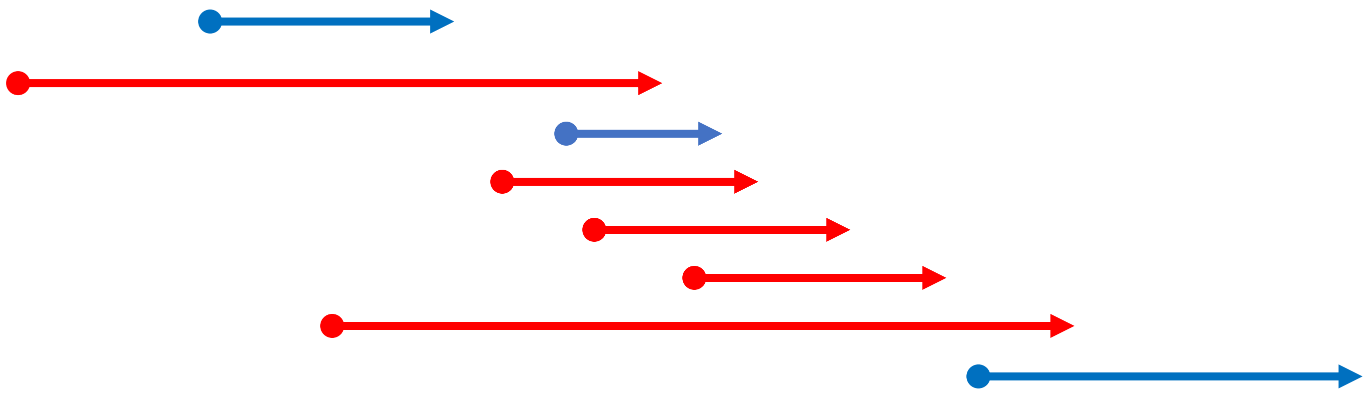
- Show exchanging an item from an arbitrary optimal solution with your greedy choice makes the new solution no worse
- How to show my sandwich is at least as good as yours:
  - Show: “I can remove any item from your sandwich, and it would be no worse by replacing it with the same item from my sandwich”

Remember: Interval Scheduling Algorithm

Find event ending earliest, add to solution,

Remove it and all conflicting events,

Repeat until all events removed, return solution



Remember: Exchange Argument

Claim: earliest ending interval is always part of some optimal solution

Let $OPT_{i,j}$ be an optimal solution for time range $[i, j]$

Let a^* be the first interval in $[i, j]$ to finish overall (greedy choice)

If $a^* \in OPT_{i,j}$ then claim holds

Else if $a^* \notin OPT_{i,j}$, let a be the first interval to end in $OPT_{i,j}$

- By definition a^* ends before a, and therefore does not conflict with any other events in $OPT_{i,j}$
- Therefore $OPT_{i,j} - \{a\} + \{a^*\}$ is also an optimal solution (same number events)
- Thus claim holds

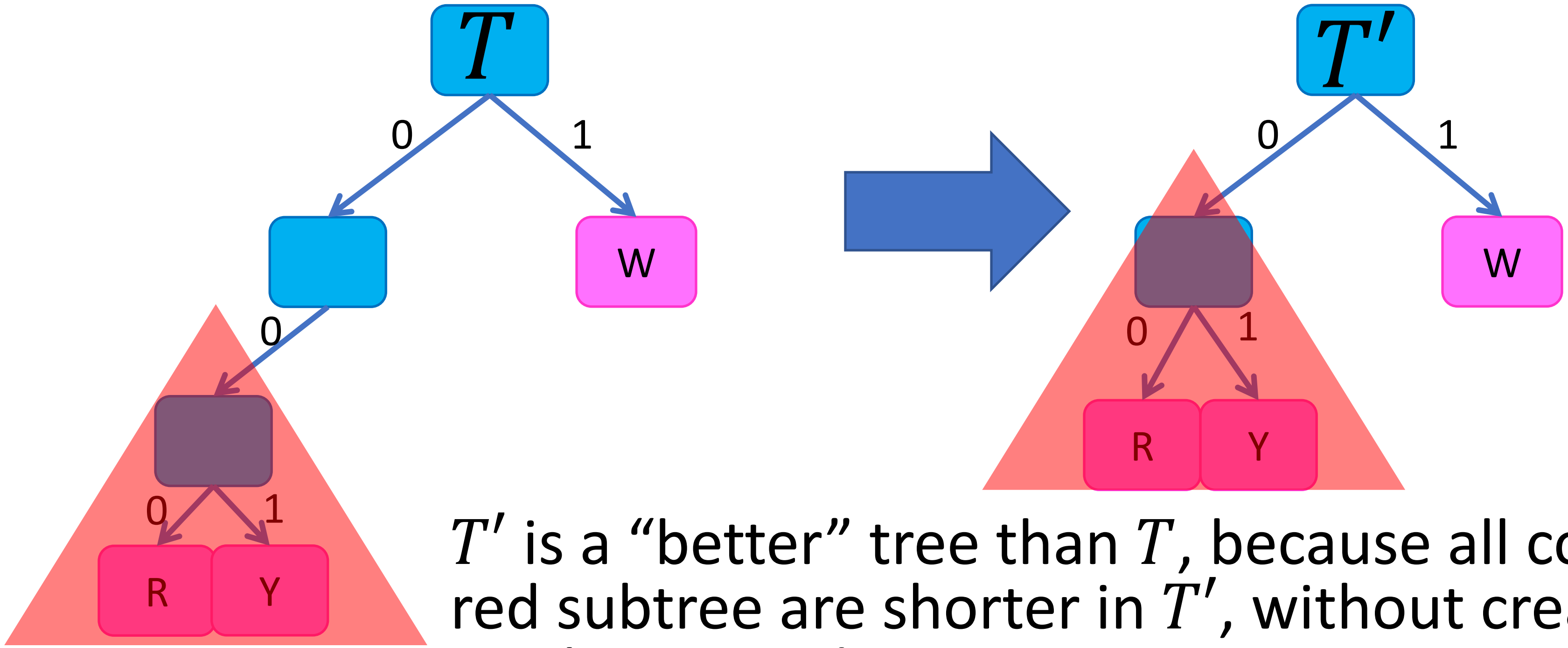
Showing Huffman is Optimal

Overview:

- Show that there is **an** optimal tree in which the least frequent characters are siblings
  - Exchange argument
- Show that making them siblings and solving the new smaller sub-problem results in **an** optimal solution
  - Optimal Substructure argument

Showing Huffman is Optimal

First Step: Show any optimal tree is “full” (each node has either 0 or 2 children)



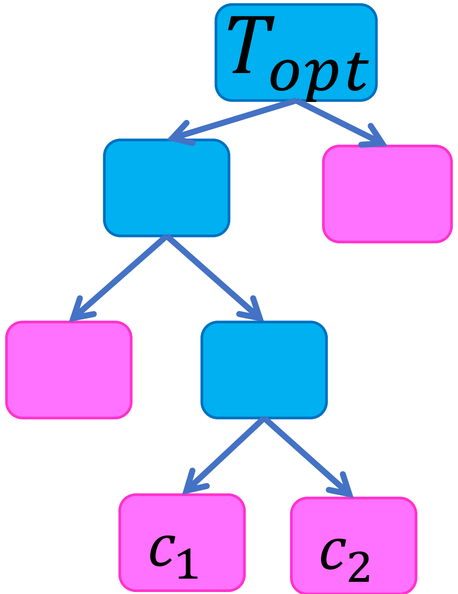
T' is a “better” tree than T, because all codes in red subtree are shorter in T', without creating any longer codes

Huffman Exchange Argument

Claim: if c_1, c_2 are the least-frequent characters, then there is an optimal prefix-free code s.t. c_1, c_2 are siblings

- i.e. codes for c_1, c_2 are the same length and differ only by their last bit

Case 1: Consider some optimal tree T_{opt}. If c_1, c_2 are siblings in this tree, then claim holds

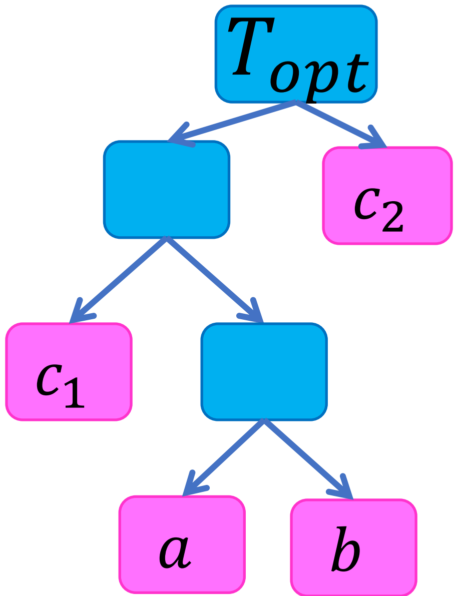


Huffman Exchange Argument

Claim: if c_1, c_2 are the least-frequent characters, then there is an optimal prefix-free code s.t. c_1, c_2 are siblings

- i.e. codes for c_1, c_2 are the same length and differ only by their last bit

Case 2: Consider some optimal tree T_{opt}, in which c_1, c_2 are not siblings



Let a, b be the two characters of lowest depth that are siblings
(Why must they exist?)

Idea: show that swapping c_1 with a does not increase cost of the tree.
Similar for c_2 and b
Assume: $f_{c1} \leq f_a$ and $f_{c2} \leq f_b$

Case 2: c_1, c_2 are not siblings in T_{opt}

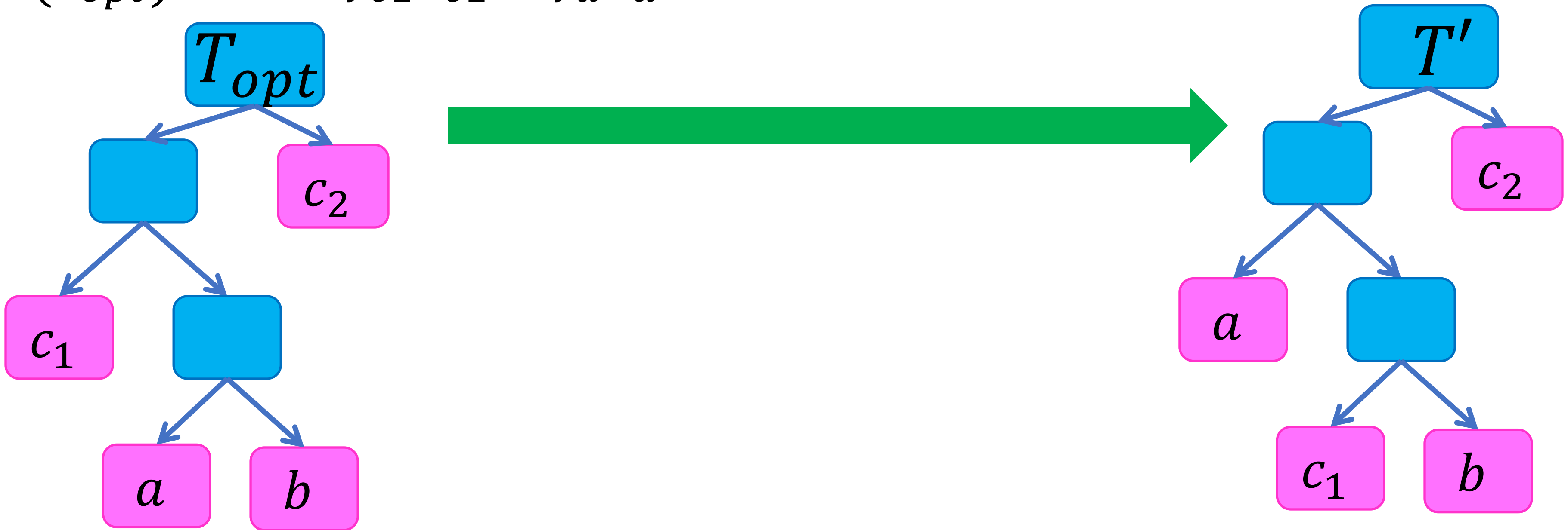
- Claim: the least-frequent characters (c_1, c_2), are siblings in some optimal tree

$a, b =$ lowest-depth siblings

Idea: show that swapping c_1 with a does not increase cost of the tree.
Assume: $f_{c1} \leq f_a$

$B(T_{opt}) = C + f_{c1}\ell_{c1} + f_a\ell_a$

$B(T') = C + f_{c1}\ell_a + f_a\ell_{c1}$



Case 2: c_1, c_2 are not siblings in T_{opt}

- Claim: the least-frequent characters (c_1, c_2), are siblings in some optimal tree

a, b = lowest-depth siblings

Idea: show that swapping c_1 with a does not increase cost of the tree.
Assume: $f_{c1} \leq f_a$

$$B(T_{opt}) = C + f_{c1}\ell_{c1} + f_a\ell_a$$

$$B(T') = C + f_{c1}\ell_a + f_a\ell_{c1}$$

$\geq 0 \Rightarrow T'$ optimal

$$B(T_{opt}) - B(T') = C + f_{c1}\ell_{c1} + f_a\ell_a - (C + f_{c1}\ell_a + f_a\ell_{c1})$$
$$= f_{c1}\ell_{c1} + f_a\ell_a - f_{c1}\ell_a - f_a\ell_{c1}$$
$$= f_{c1}(\ell_{c1} - \ell_a) + f_a(\ell_a - \ell_{c1})$$
$$= (f_a - f_{c1})(\ell_a - \ell_{c1})$$

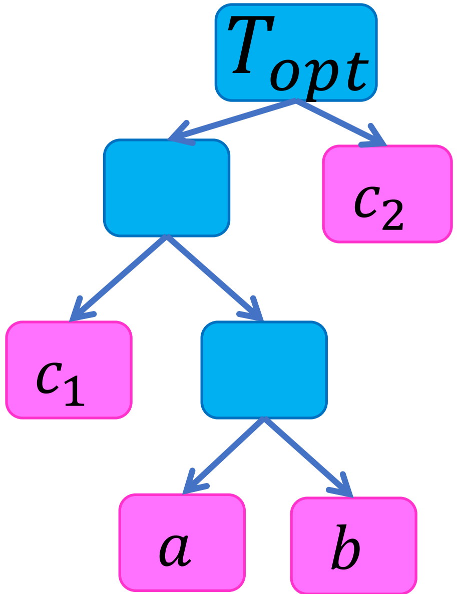
Case 2: c_1, c_2 are not siblings in T_{opt}

- Claim: the least-frequent characters (c_1, c_2), are siblings in some optimal tree

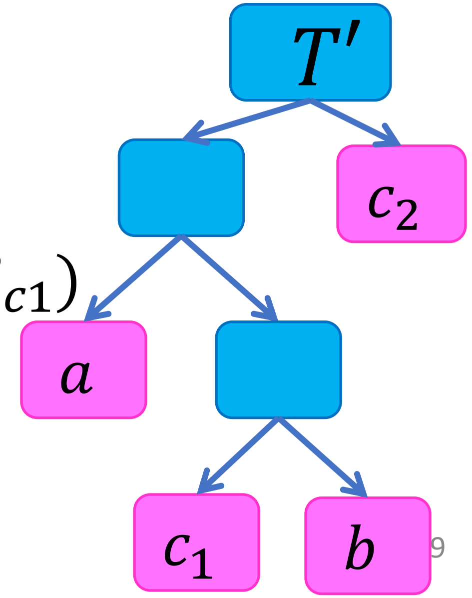
a, b = lowest-depth siblings

Idea: show that swapping c_1 with a does not increase cost of the tree.
Assume: $f_{c1} \leq f_a$

$$B(T_{opt}) = C + f_{c1}\ell_{c1} + f_a\ell_a$$



$$B(T') = C + f_{c1}\ell_a + f_a\ell_{c1}$$



$$B(T_{opt}) - B(T') = (f_a - f_{c1})(\ell_a - \ell_{c1})$$

≥ 0 ≥ 0

$$B(T_{opt}) - B(T') \geq 0$$

T' is also optimal!

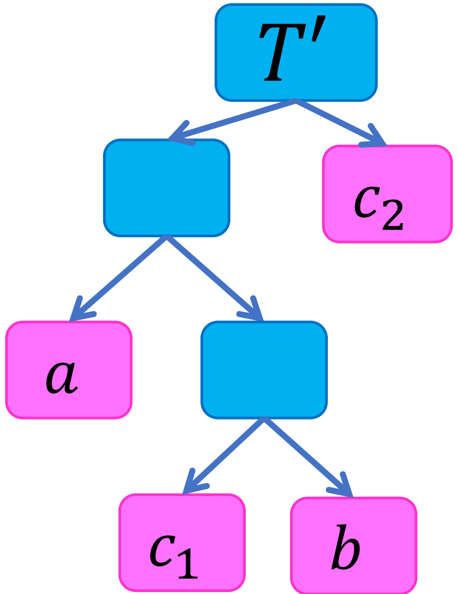
Case 2: Repeat to swap c_2, b!

- Claim: the least-frequent characters (c_1, c_2), are siblings in some optimal tree

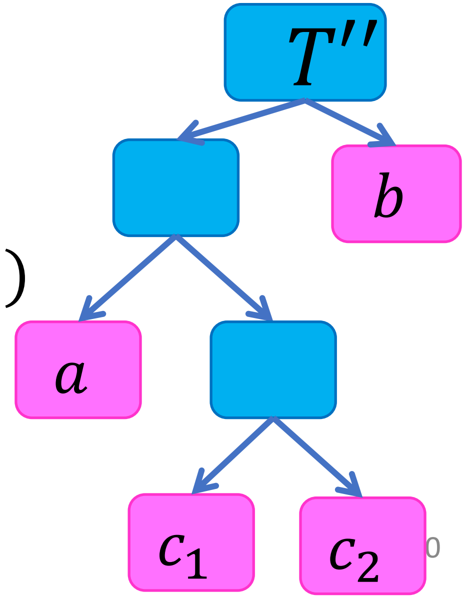
$a, b =$ lowest-depth siblings

Idea: show that swapping c_2 with b does not increase cost of the tree.
Assume: $f_{c2} \leq f_b$

$B(T') = C + f_{c2}\ell_{c2} + f_b\ell_b$



$B(T'') = C + f_{c2}\ell_b + f_b\ell_{c2}$



$B(T') - B(T'') = (f_b - f_{c2})(\ell_b - \ell_{c2})$

≥ 0 $\quad$ ≥ 0

$B(T') - B(T'') \geq 0$

T'' is also optimal! Claim holds!

Showing Huffman is Optimal

Overview:

- Show that there is **an** optimal tree in which the least frequent characters are siblings
    - Exchange argument
- Show that making them siblings and solving the new smaller sub-problem results in **an** optimal solution
    - Optimal Substructure argument

Proving Optimal Substructure

Goal: show that if x is in an optimal solution, then the rest of the solution is an optimal solution to the subproblem.

Usually by Contradiction:

- Assume that x must be an element of my optimal solution
- Assume that solving the subproblem induced from choice x, then adding in x is not optimal
- Show that removing x from a better overall solution must produce a better solution to the subproblem

Huffman Optimal Substructure

Goal: show that if c_1, c_2 are siblings in an optimal solution, then an optimal prefix free code can be found by using a new character with frequency $f_{c_1} + f_{c_2}$ and then making c_1, c_2 its children.

By Contradiction:

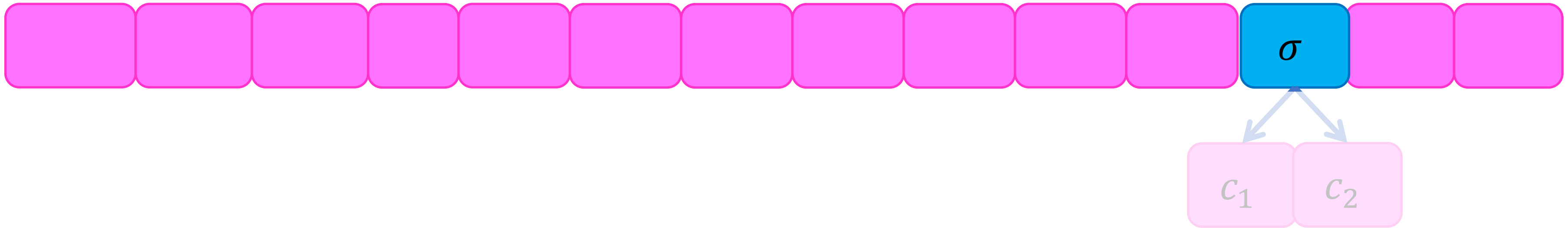
- Assume that c_1, c_2 are siblings in at least one optimal solution
- Assume that solving the subproblem with this new character, then adding in c_1, c_2 is not optimal
- Show that removing c_1, c_2 from a better overall solution must produce a better solution to the subproblem

Finishing the Proof

Show Recursive Substructure

- Show treating c_1, c_2 as a new "combined" character gives optimal solution

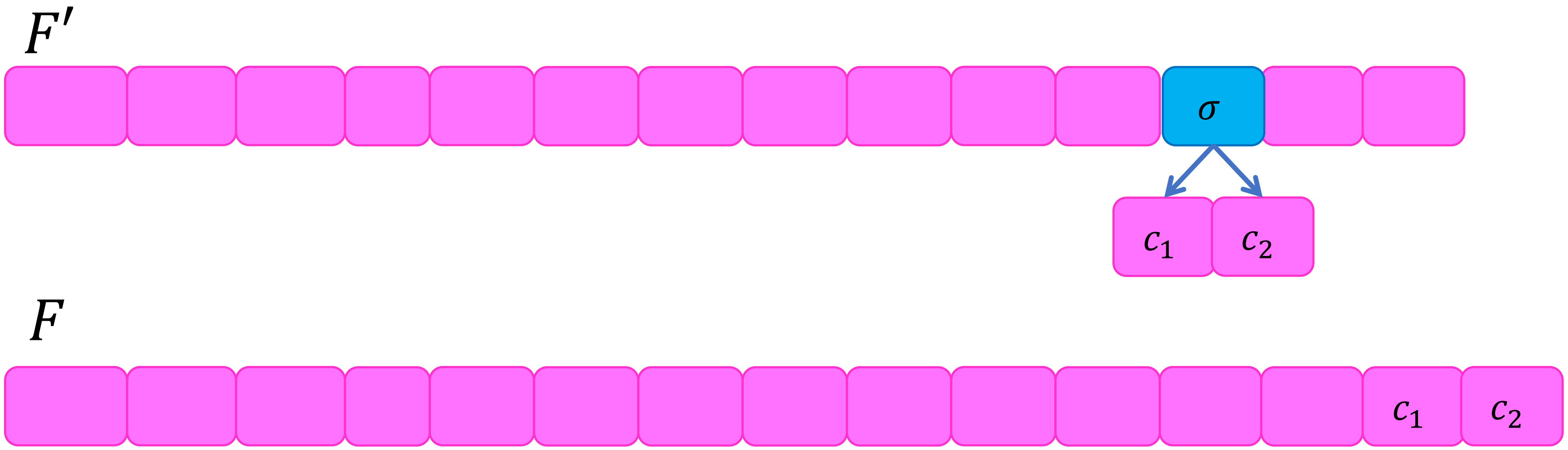
Why does solving this smaller problem:



Give an optimal solution to this?:

Substructure

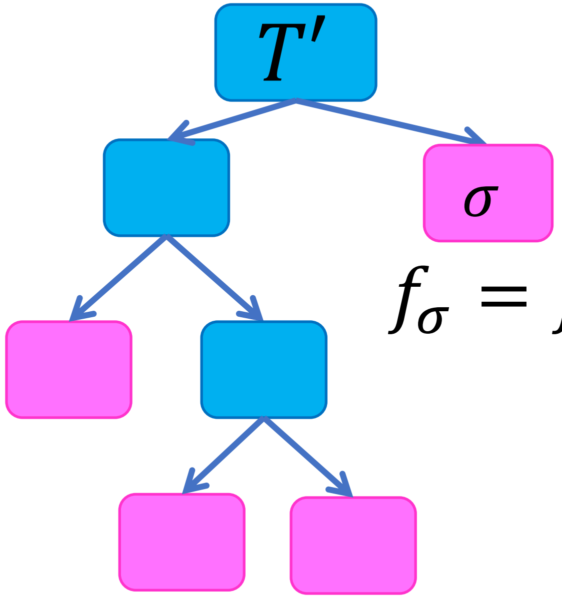
Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ



Substructure

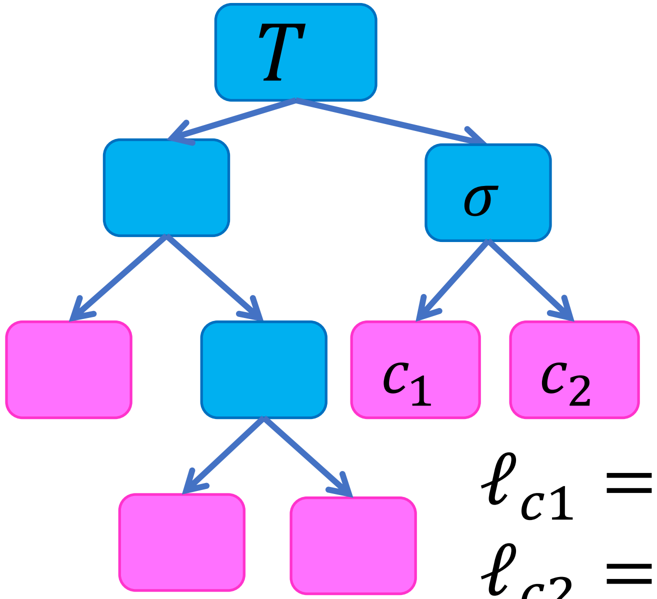
Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ

If this is optimal



$$f_\sigma = f_{c1} + f_{c2}$$

Then this is optimal



$$\ell_{c1} = \ell_\sigma + 1$$
$$\ell_{c2} = \ell_\sigma + 1$$

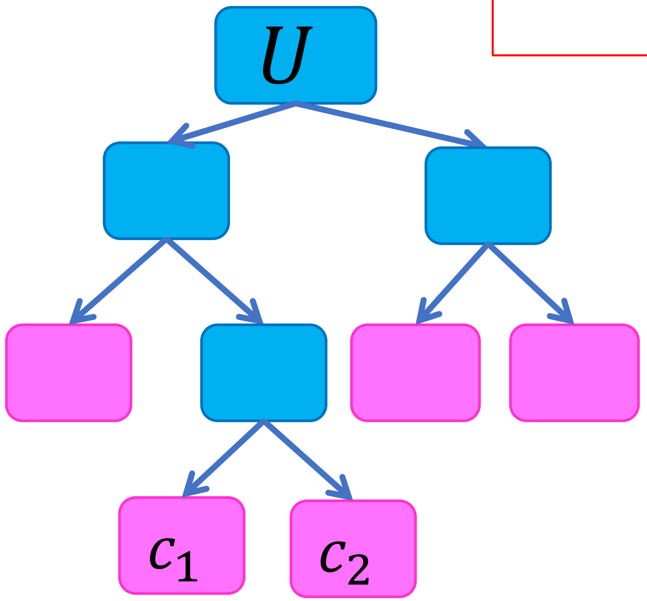
$$B(T') = B(T) - f_{c1} - f_{c2}$$

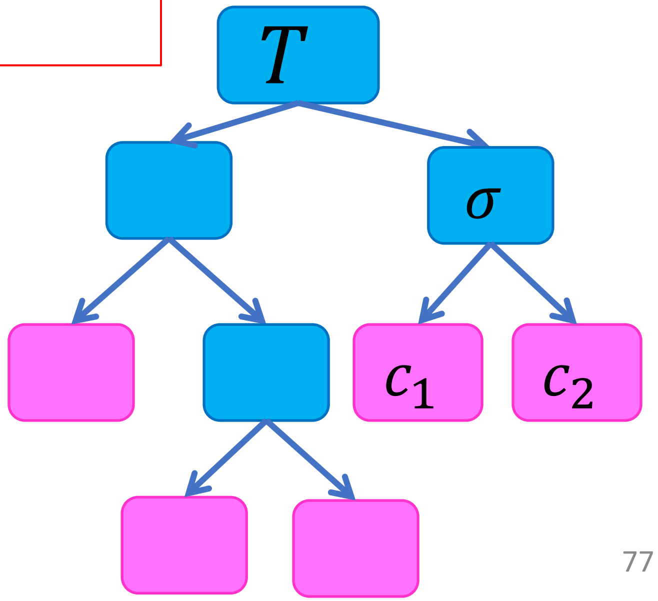
Substructure

Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ

Toward contradiction

Suppose T is not optimal
Let U be a lower-cost tree
$$B(U) < B(T)$$



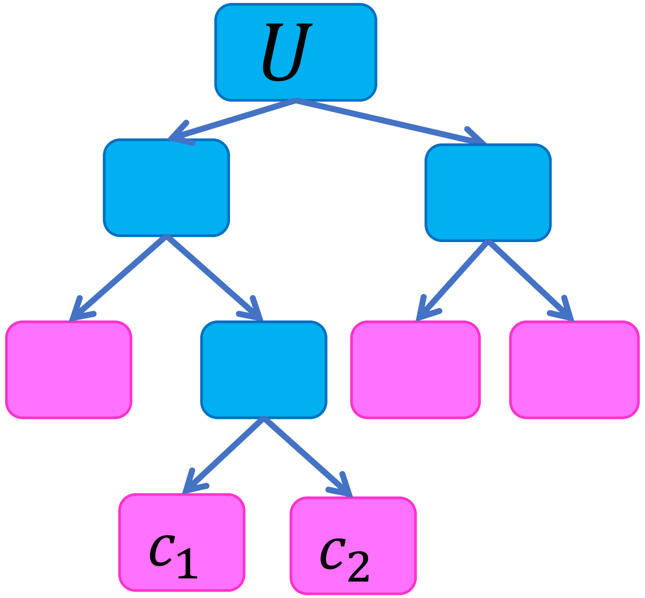


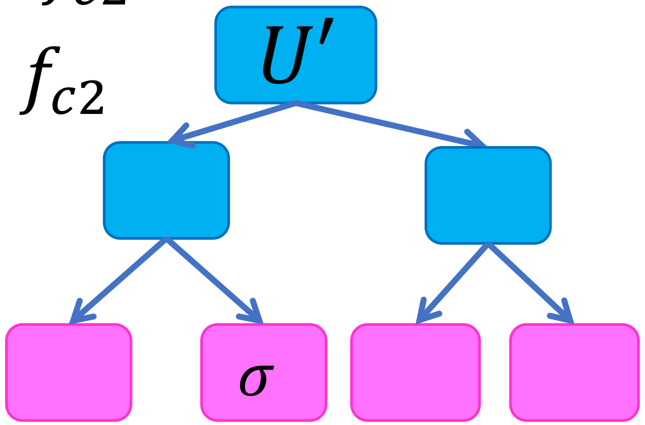
Substructure

Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ

$$B(U) < B(T)$$

$$\begin{aligned} B(U') &= B(U) - f_{c1} - f_{c2} \\ &< B(T) - f_{c1} - f_{c2} \\ &= B(T') \end{aligned}$$





Contradicts optimality of T', so T is optimal!

Optimal Substructure

Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ