CS 3100
Data Structures and Algorithms 2
Lecture 9: D&C: Closest Pair of Points

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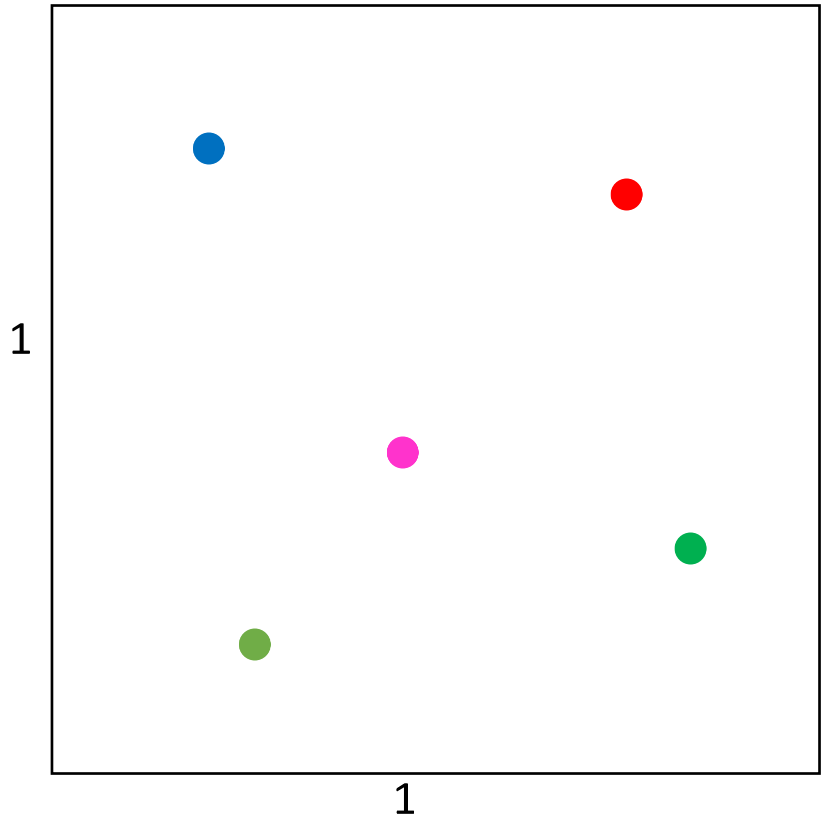
Spring 2024

Readings in CLRS 4th edition:

- Section 4.5

Warm up

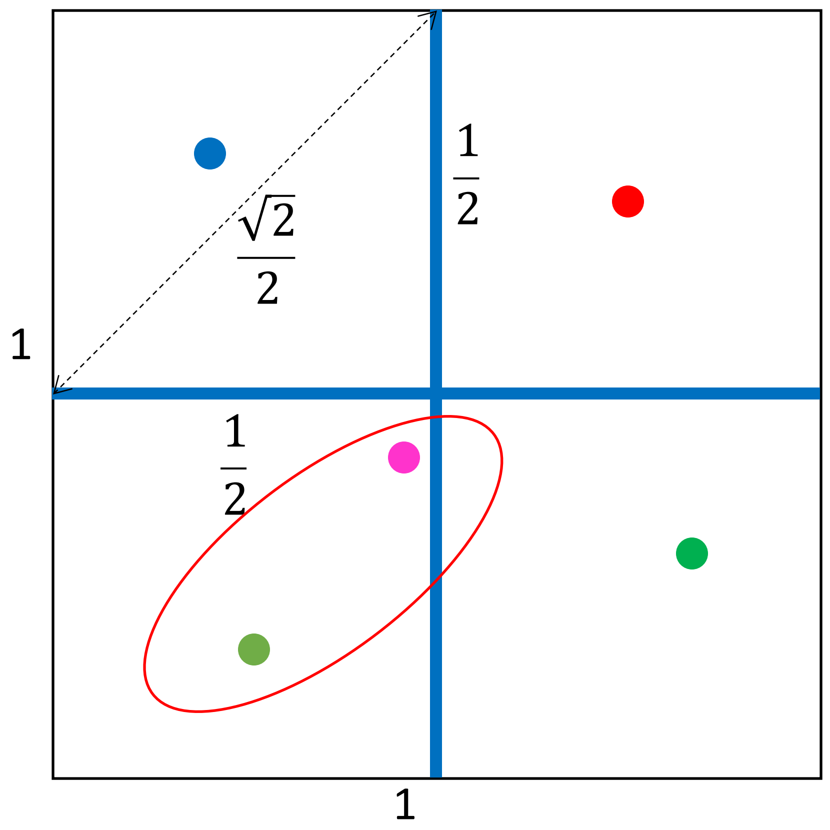
Given any 5 points on the unit square, show there's always a pair distance $\leq \frac{\sqrt{2}}{2}$ apart



If points p_1, p_2 in same quadrant, then $\delta(p_1, p_2) \leq \frac{\sqrt{2}}{2}$

Given 5 points, two must share the same quadrant

Pigeonhole Principle!



Announcements

- PS4 coming soon
- Office hours
    - Prof Hott Office Hours: Mondays 11a-12p, Fridays 10-11a and 2-3p
    - Prof Pettit Office Hours: Mondays and Wednesdays 2:30-4:00p
    - TA office hours posted on our website
- Quizzes 1-2 coming February 29, 2024
    - Both quizzes taken the same day
    - If you have SDAC, please schedule for 1 exam (*not a quiz*)

Divide and Conquer

[CLRS Chapter 4]

Divide:

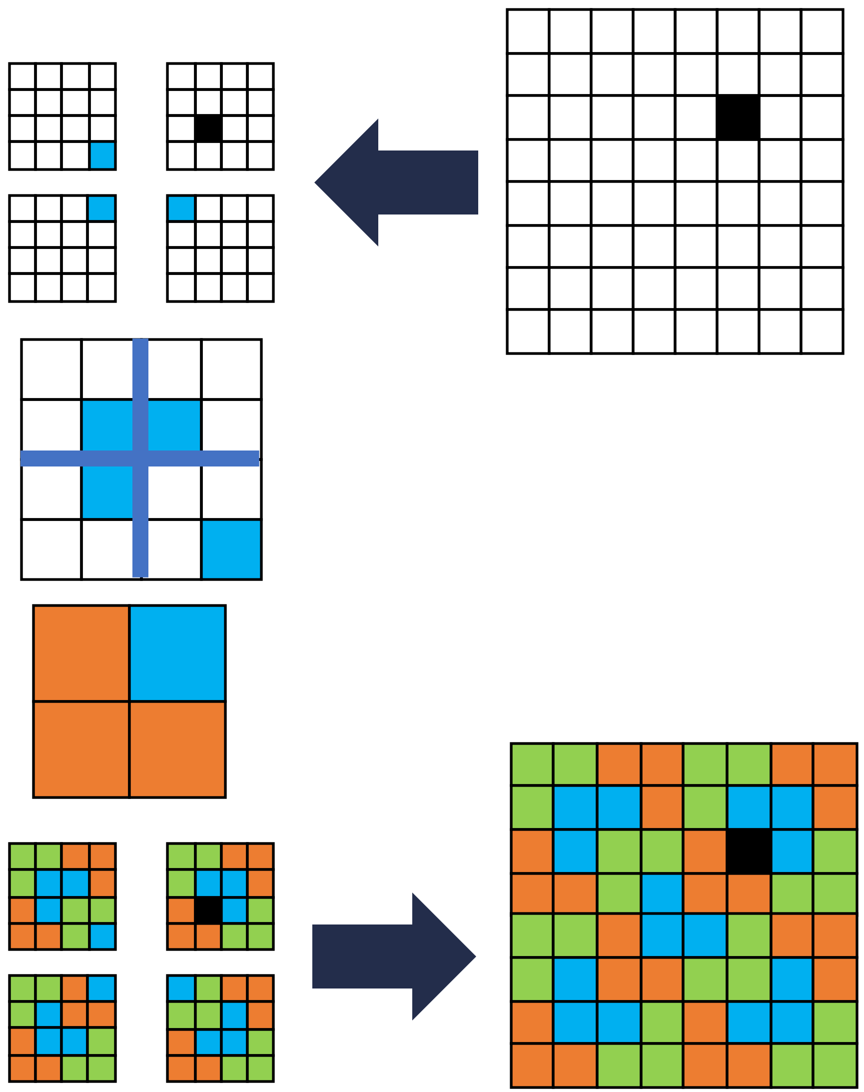
- Break the problem into multiple subproblems, each smaller instances of the original

Conquer:

- If the suproblems are "large":
  - Solve each subproblem recursively
- If the subproblems are "small":
  - Solve them directly (base case)

Combine:

- Merge solutions to subproblems to obtain solution for original problem



Divide and Conquer

[CLRS Chapter 4]

Base Case:

- If the problem is “small”, solve directly (brute force)

Divide:

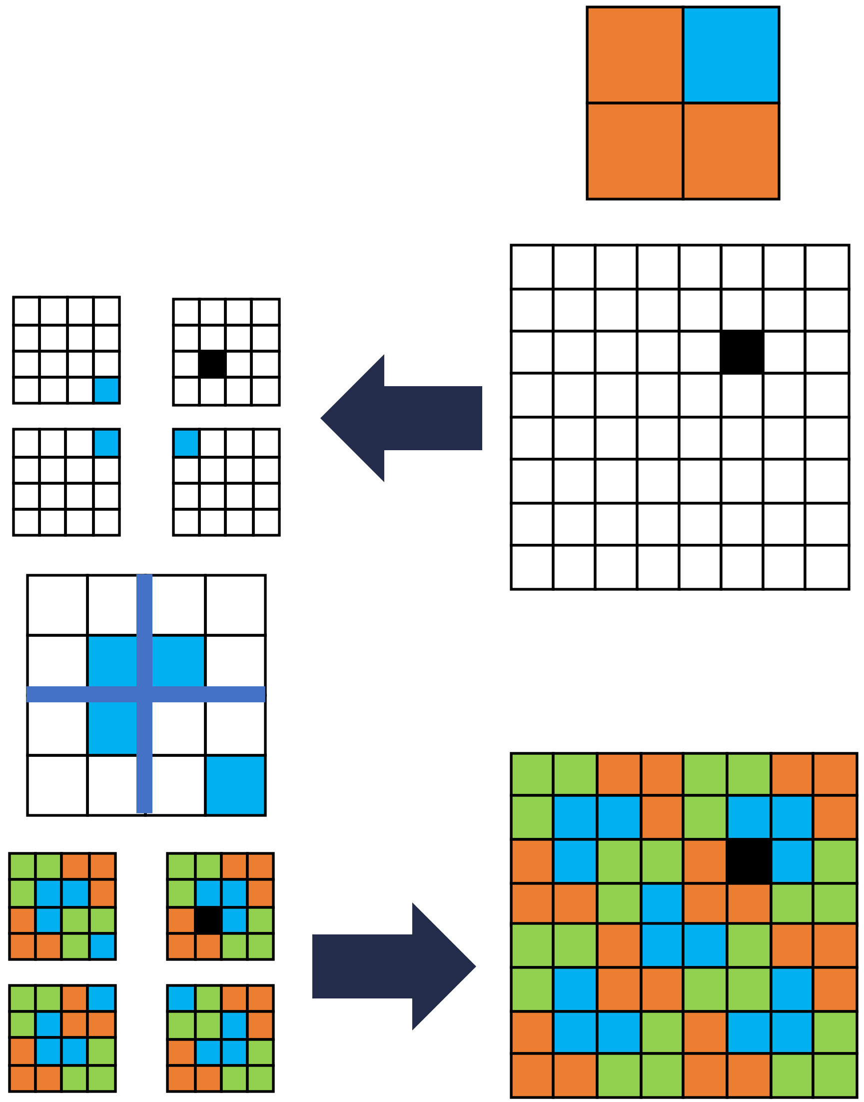
- Break the problem into multiple subproblems, each smaller instances of the original

Conquer:

- Solve each subproblem recursively

Combine:

- Merge solutions to subproblems to obtain solution for original problem



Observation

Divide: $D(n)$ time

Conquer: Recurse on smaller problems of size $s_1, \ldots, s_k$

Combine: $C(n)$ time

Recurrence:

- $T(n) = \underline{D(n)} + \underbrace{\sum_{i \in [k]} T(s_i)}_{\text{cost of tree below}} + \underline{C(n)}$

Many divide and conquer algorithms have recurrences are of form:

- $T(n) = a \cdot T(n/b) + f(n)$

a and b are constants

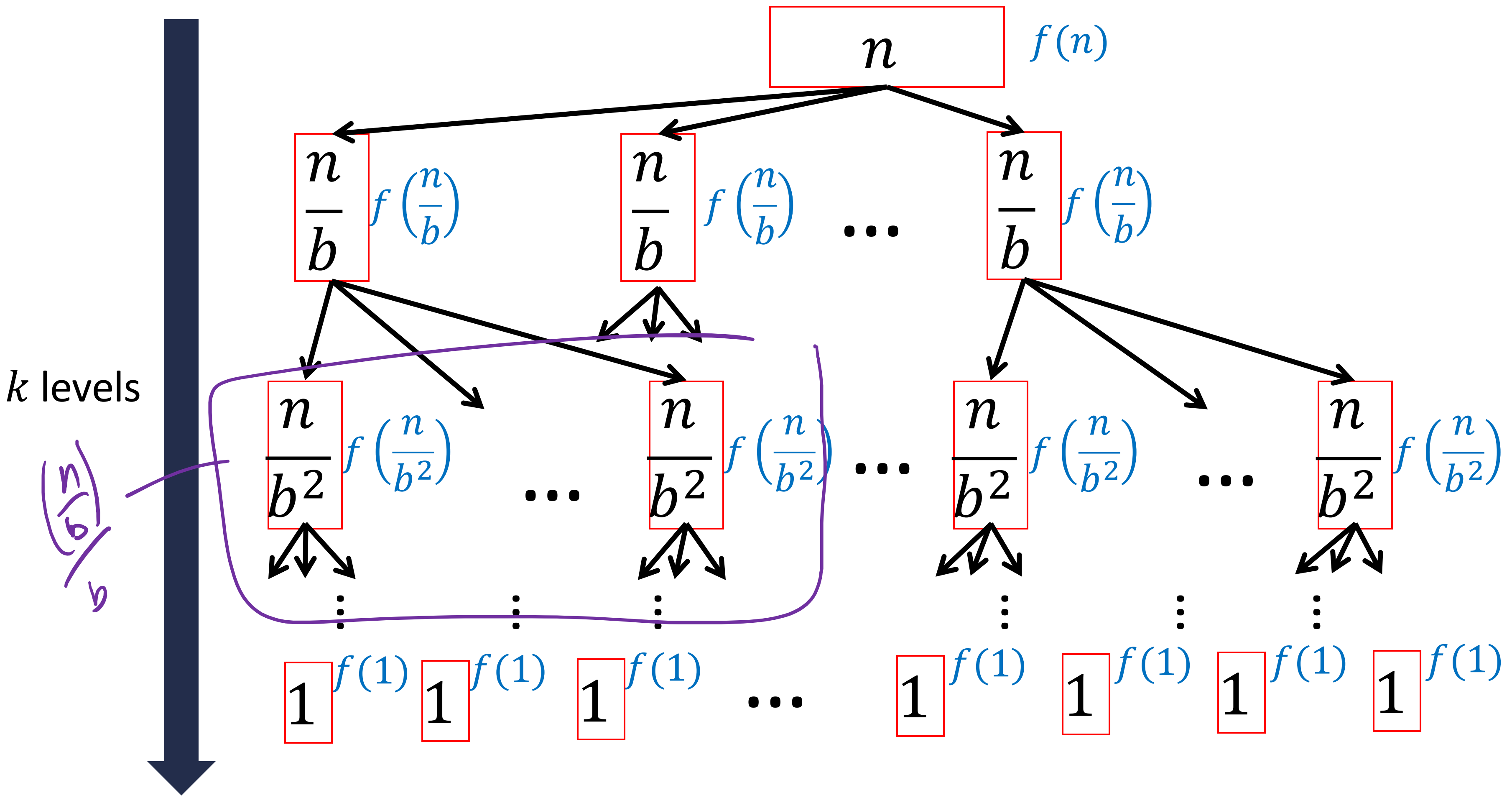
Mergesort: $T(n) = 2T(n/2) + n$

Divide and Conquer Multiplication: $T(n) = 4T(n/2) + 5n$

Karatsuba Multiplication: $T(n) = 3T(n/2) + 8n$

General Recurrence

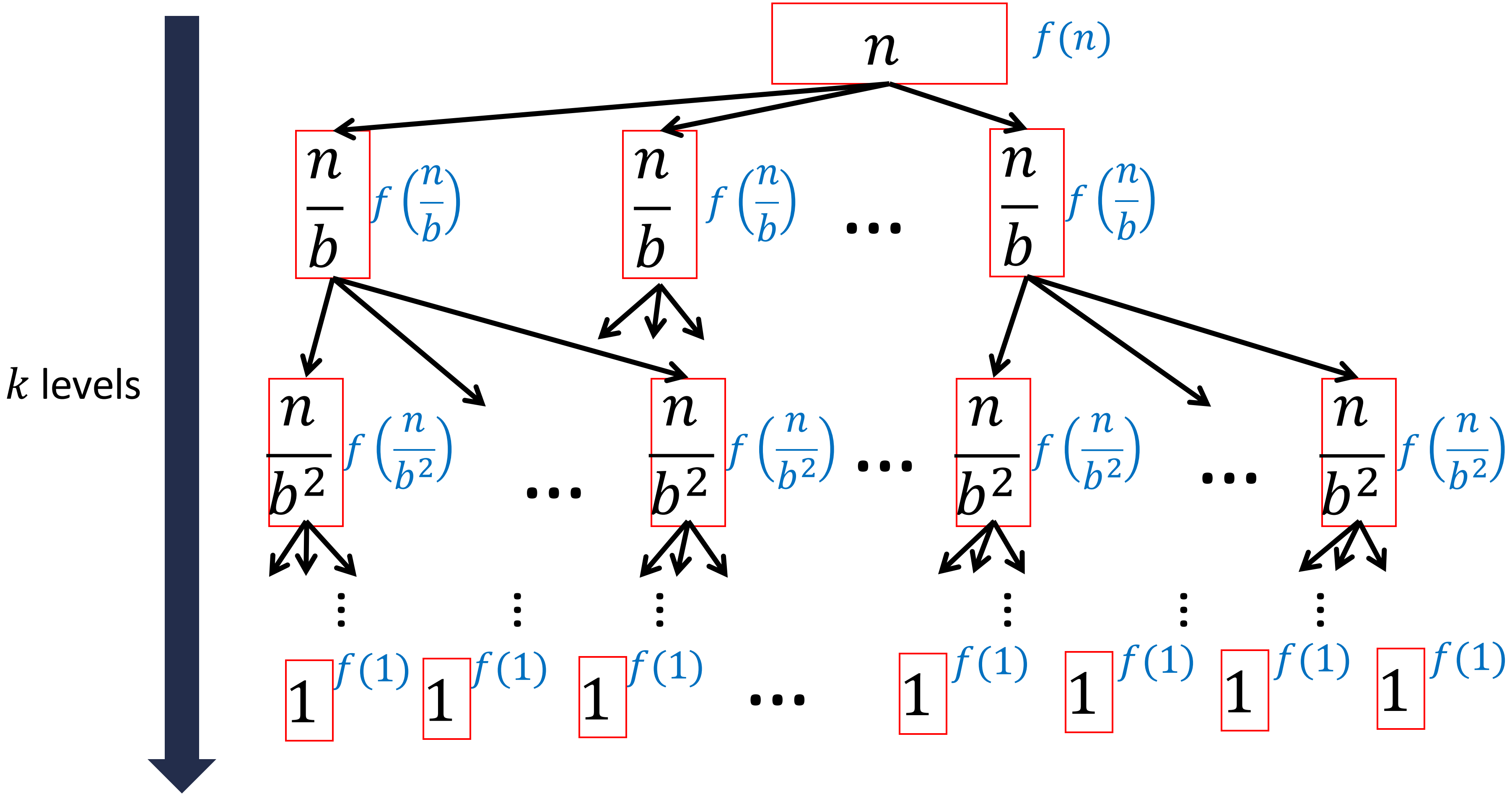
$$T(n) = aT(n/b) + f(n)$$



Number of subproblems	Cost of subproblem
1	$f(n)$
a	$f(n/b)$
a^2	$f(n/b^2)$
a^k	$f(n/b^k)$

General Recurrence

$$T(n) = {\color{red}a}T(n/b) + {\color{blue}f(n)}$$



Number of subproblems	Cost of subproblem
1	$f(n)$
a	$f(n/b)$
a^2	$f(n/b^2)$
$\boxed{a^{\log_b n} = n^{\log_b a}}$	$f(n/b^k)$

$f(n) \overset{?}{\;} n^{\log_b a}$

Three Cases

$\Theta(n^{\log_b a})$

1st 2nd 3rd 4th

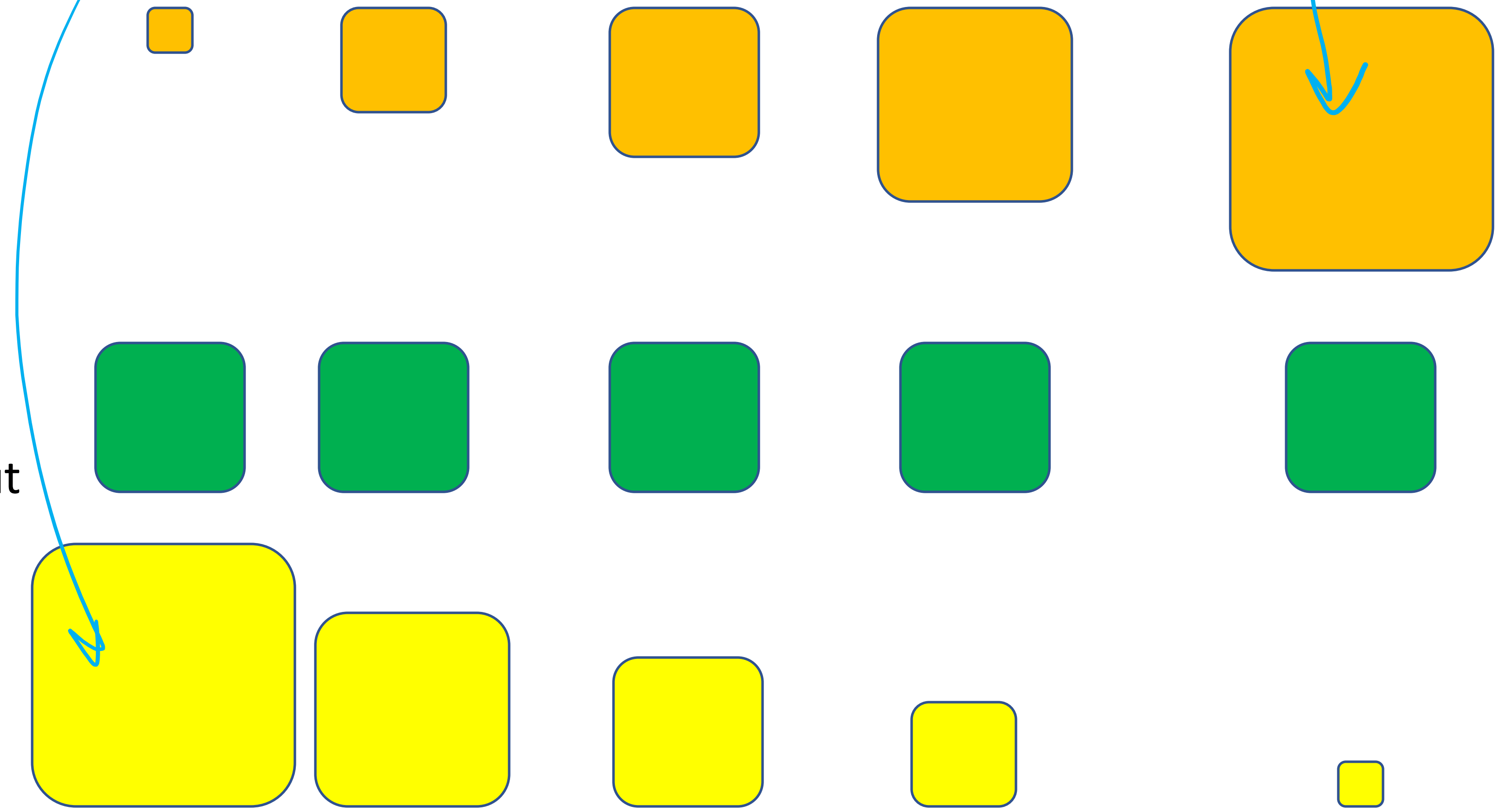
$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^k f\left(\frac{n}{b^k}\right)$$

$k = \log_b n$

Case 1:
Most work happens at the leaves

Case 2:
Work happens consistently throughout

Case 3:
Most work happens at top of tree



Master Theorem

$f(n)$? $n^{\log_b a}$

$$T(n) = aT(n/b) + f(n)$$

$$\delta = \log_b a$$

	Requirement on f	Implication
Case 1	$f(n) \in O\left(n^{\delta-\varepsilon}\right)$ for some constant $\varepsilon > 0$	$T(n) \in \Theta\left(n^{\delta}\right)$

Master Theorem

$$T(n) = aT(n/b) + f(n)$$

$$\delta = \log_b a$$

	Requirement on f	Implication
Case 1	$f(n) \in O\left(n^{\delta-\varepsilon}\right)$ for some constant $\varepsilon > 0$	$T(n) \in \Theta\left(n^{\delta}\right)$
Case 2	$f(n) \in \Theta\left(n^{\delta}\right)$	$T(n) \in \Theta\left(n^{\delta} \log n\right)$

Master Theorem

$$T(n) = aT(n/b) + f(n)$$

$$\delta = \log_b a$$

	Requirement on f	Implication
Case 1	$f(n) \in O\left(n^{\delta-\varepsilon}\right)$ for some constant $\varepsilon > 0$	$T(n) \in \Theta\left(n^{\delta}\right)$
Case 2	$f(n) \in \Theta\left(n^{\delta}\right)$	$T(n) \in \Theta\left(n^{\delta} \log n\right)$
Case 3	$f(n) \in \Omega\left(n^{\delta+\varepsilon}\right)$ for some constant $\varepsilon > 0$ 2nd level top **AND** $af\left(\frac{n}{b}\right) \leq cf(n)$ for constant $c < 1$ and sufficiently large n	$T(n) \in \Theta\left(f(n)\right)$

Master Theorem Example 1

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$a=2 \quad b=2$

$n^{\log_2 2} = n$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$\underline{f(n)} \quad \underline{n^{\log_b a}}$

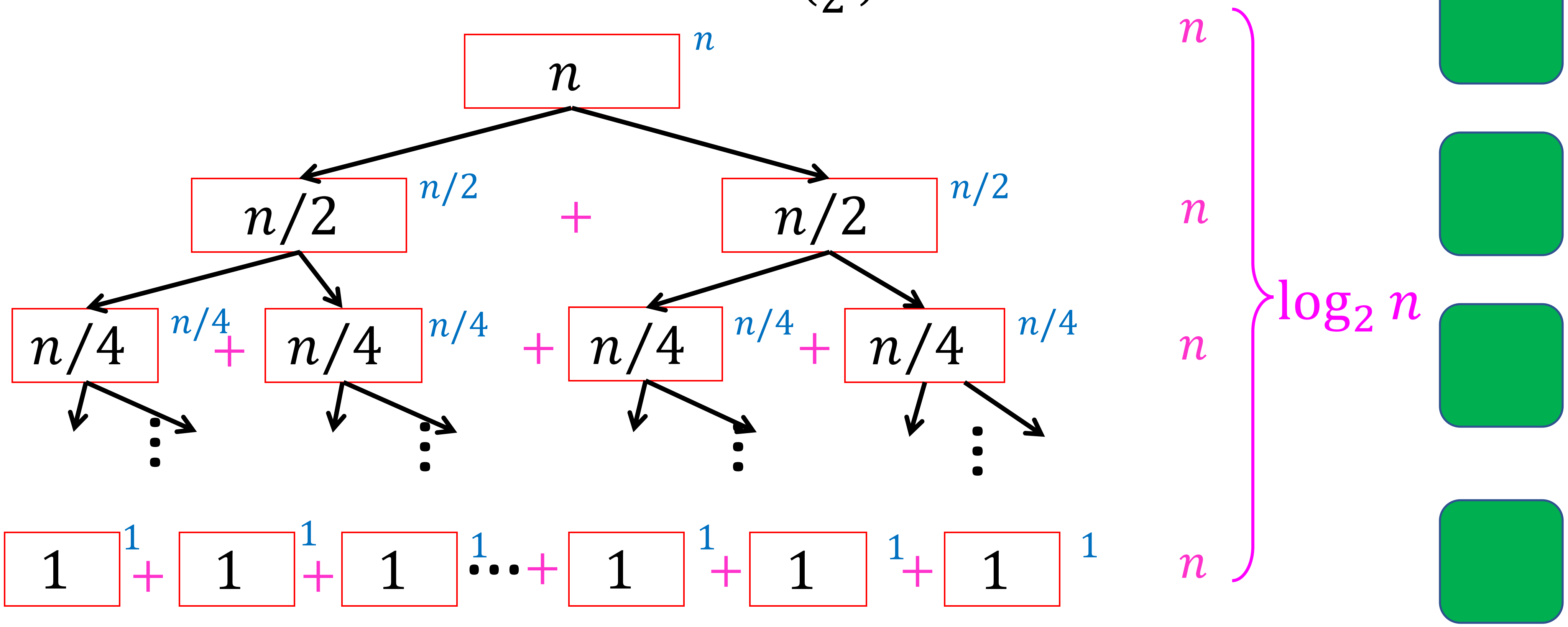
$n \quad \in \Theta(n)$

Case 2

$$\Theta\left(n^{\log_2 2} \log n\right) = \Theta(n \log n)$$

Tree method

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



Master Theorem Example 2

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$a = 4,\ b = 2$

$n^{\log_2 4} = n^2$

$$T(n) = 4T\left(\frac{n}{2}\right) + \underbrace{5n}_{}$$

D+C

$\frac{f(n)}{5n} \quad \frac{n^{\log_b a}}{\in O(n^2)}$

$\varepsilon = 1$

$n^{\log_b a - \varepsilon}$

$5n \in O(n)$

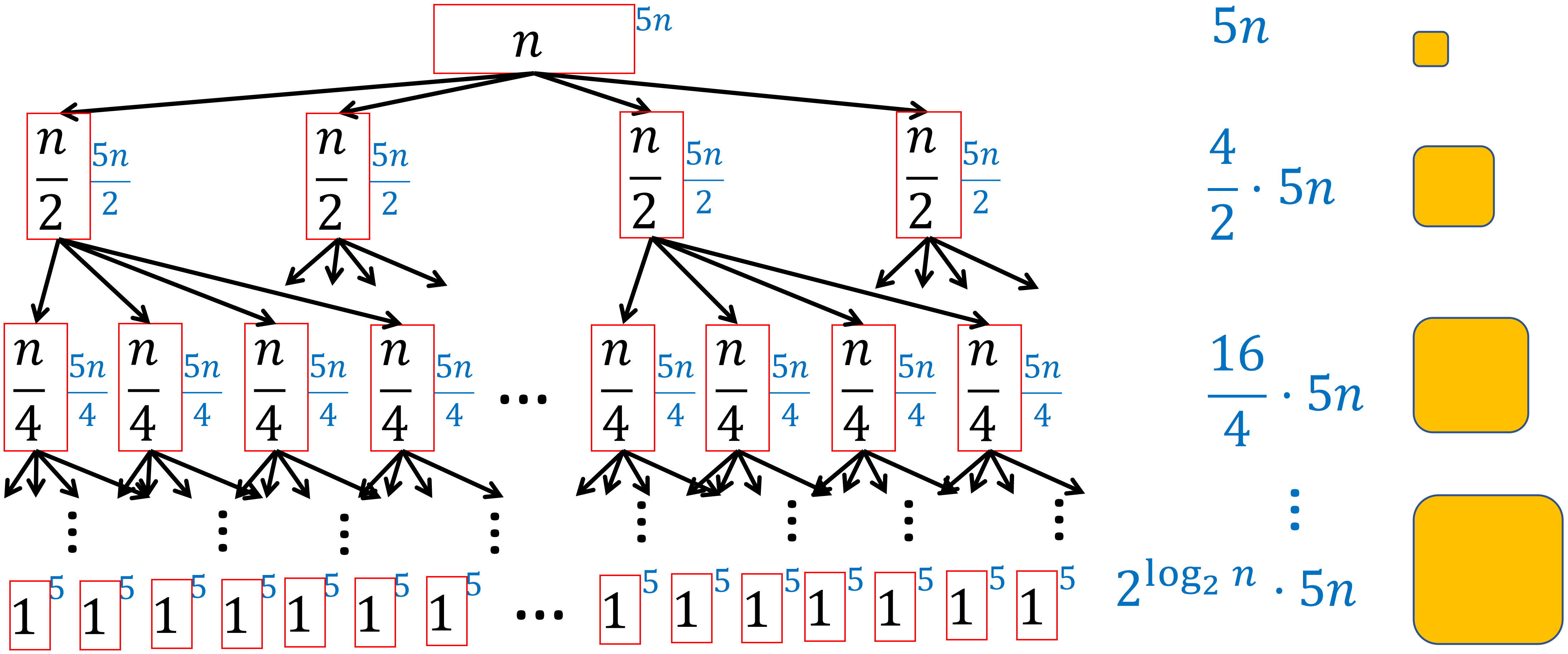
$n^{2-1} = n$

Case 1

$$\Theta(n^{\log_2 4}) = \Theta(n^2)$$

Tree method

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$



Tree method

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

Cost is <u>increasing</u> with the recursion depth (due to large number of subproblems)

Most of the work happening in the leaves

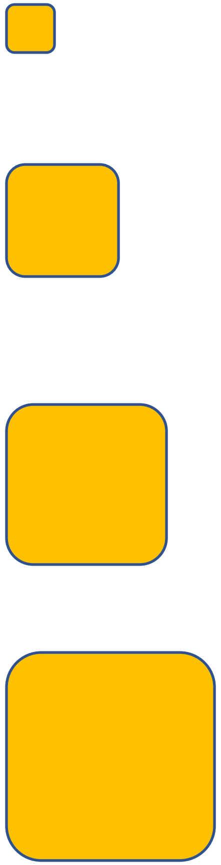
$5n$

$\frac{4}{2} \cdot 5n$

$\frac{16}{4} \cdot 5n$

$\vdots$

$2^{\log_2 n} \cdot 5n$



Master Theorem Example 3

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$a = 3,\ b = 2$

$n^{\log_2 3} =$

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$\underline{f(n)}$ $\quad \underline{n^{\log_b a}}$

$8n \in O\left(n^{\log_2 3 - 0.3}\right)$

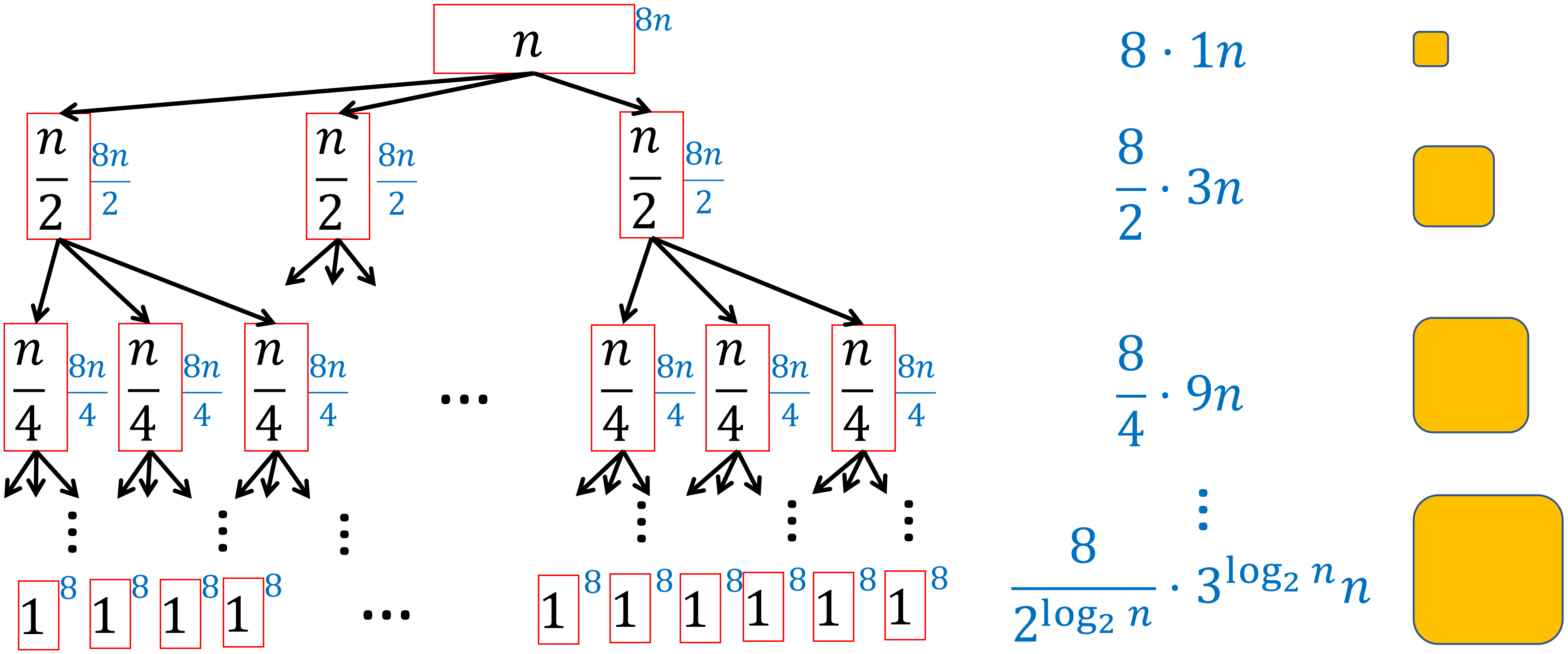
$\varepsilon = 0.3$

Case 1

$$\Theta\left(n^{\log_2 3}\right) \approx \Theta(n^{1.585})$$

Karatsuba

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$



Master Theorem Example 4

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$a = 2,\ b = 2$

$n^{\log_2 2} = n$

$\boxed{D(n) + C(n)}$

$$T(n) = 2T\left(\frac{n}{2}\right) + 15n^3$$

$\frac{f(n)}{15n^3} \in \frac{n^{\log_b a}}{\Omega(n^{1+1}) = \Omega(n^2)}$

$\varepsilon = 1$

Case 3

Master Theorem Example 4

$2f(\frac{n}{2})$
$= 2 \cdot 15 (\frac{n}{2})^3$
$= \frac{30}{\cancel{8}} n^3$
$= \frac{15}{4} n^3$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$$T(n) = 2T\left(\frac{n}{2}\right) + 15n^3$$

Case 3

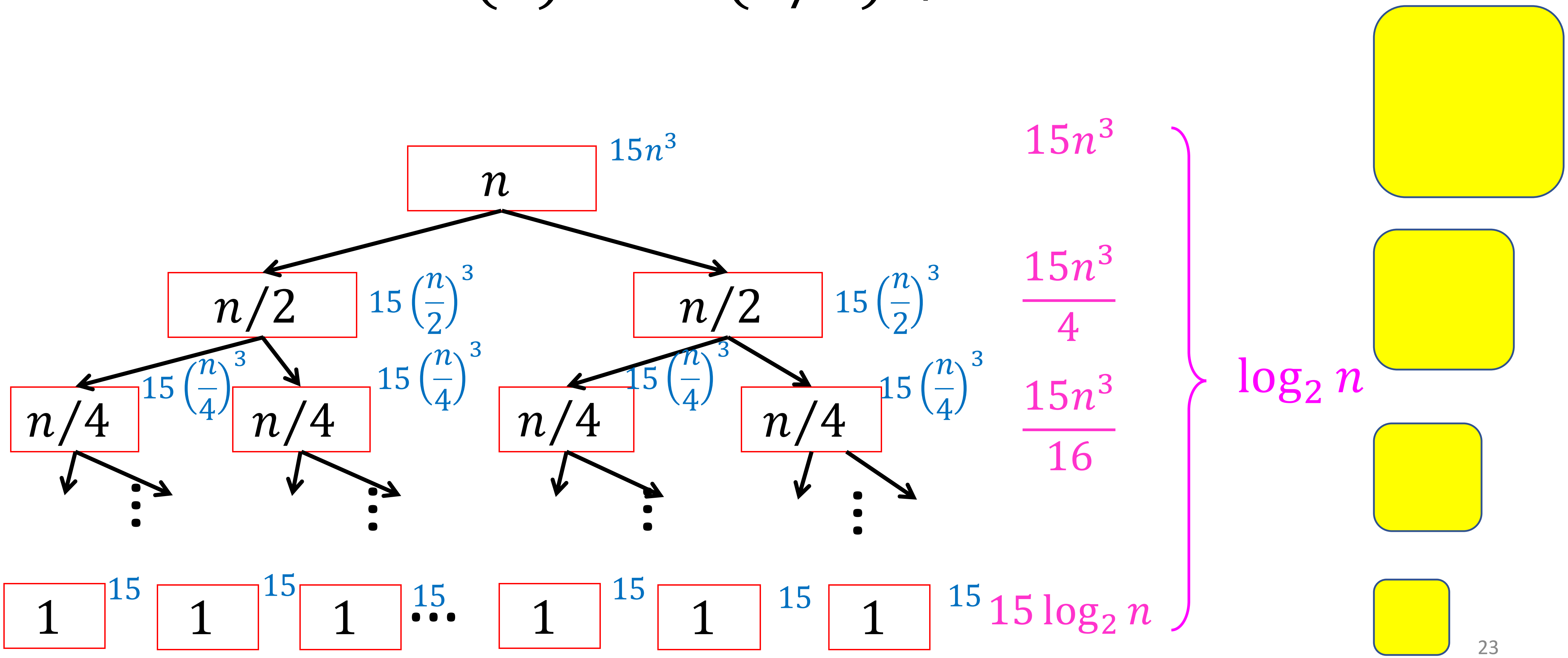
$\Theta(n^3)$

Important: For Case 3, need to additionally check that $2f(n/2) \le cf(n)$ for constant $c < 1$ and sufficiently large n

$$2f(n/2) = 30(n/2)^3 = \frac{\overset{15}{\cancel{30}}}{\underset{4}{\cancel{8}}} n^3 \le \overset{c}{\frac{1}{4}}(15n^3)$$

Master Theorem Example 4 (Visually)

$$T(n) = 2T(n/2) + 15n^3$$



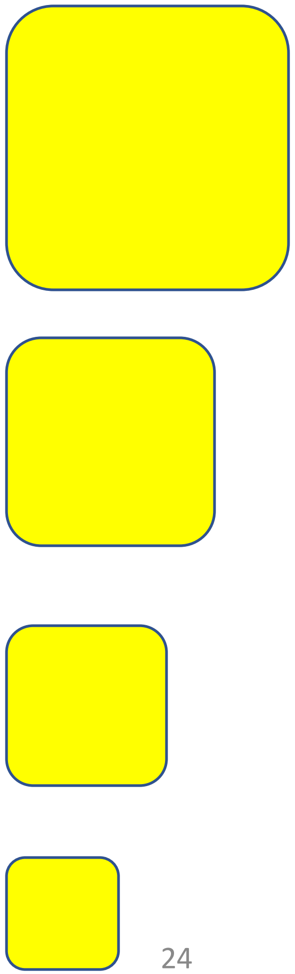
Master Theorem Example 4 (Visually)

$$T(n) = 2T(n/2) + 15n^3$$

Cost is <u>decreasing</u> with the recursion depth (due to high *non-recursive* cost)

Most of the work happening at the top

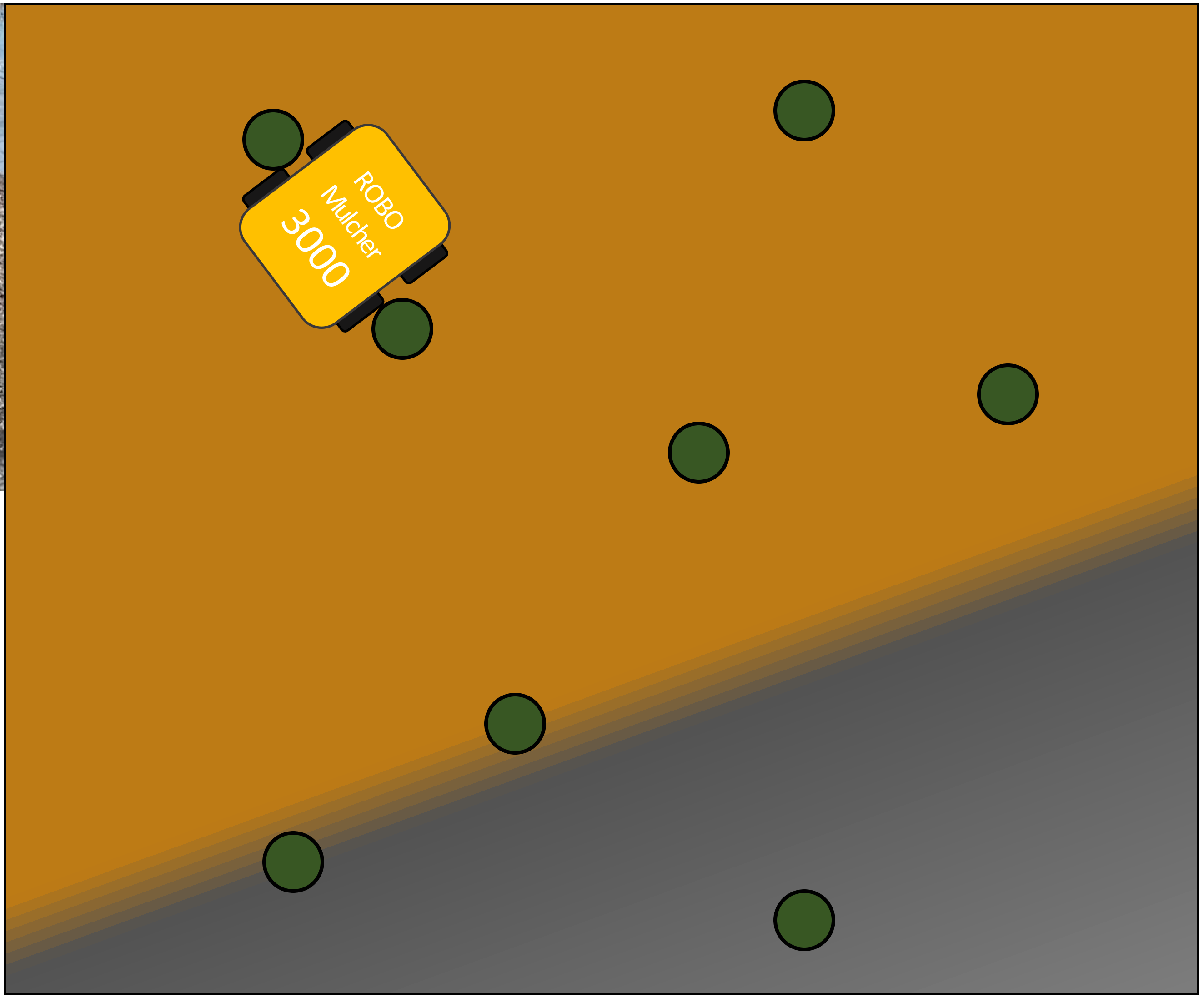
$$\left.\begin{array}{c} 15n^3 \\[1em] \dfrac{15n^3}{4} \\[1em] \dfrac{15n^3}{16} \\[2em] 15\log_2 n \end{array}\right\} \log_2 n$$



Robbie’s Yard

There Has to be an Easier Way!

Constraints: Trees and Plants



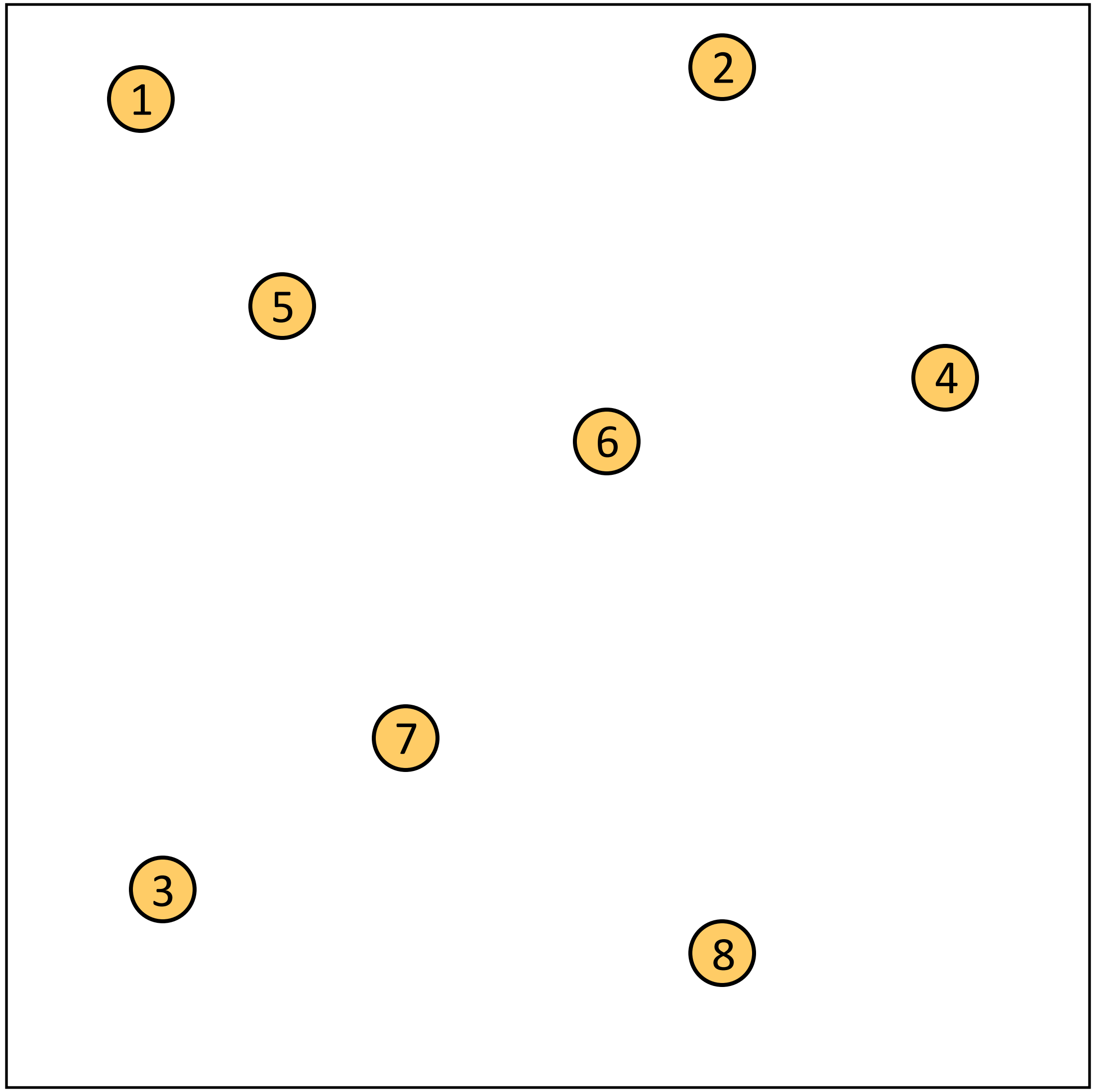
How wide can the robot be?

Objective: find closest pair of trees

Closest Pair of Points

Given: A list of points

Return: Pair of points with smallest distance apart



Closest Pair of Points: Naïve

Given: A list of points

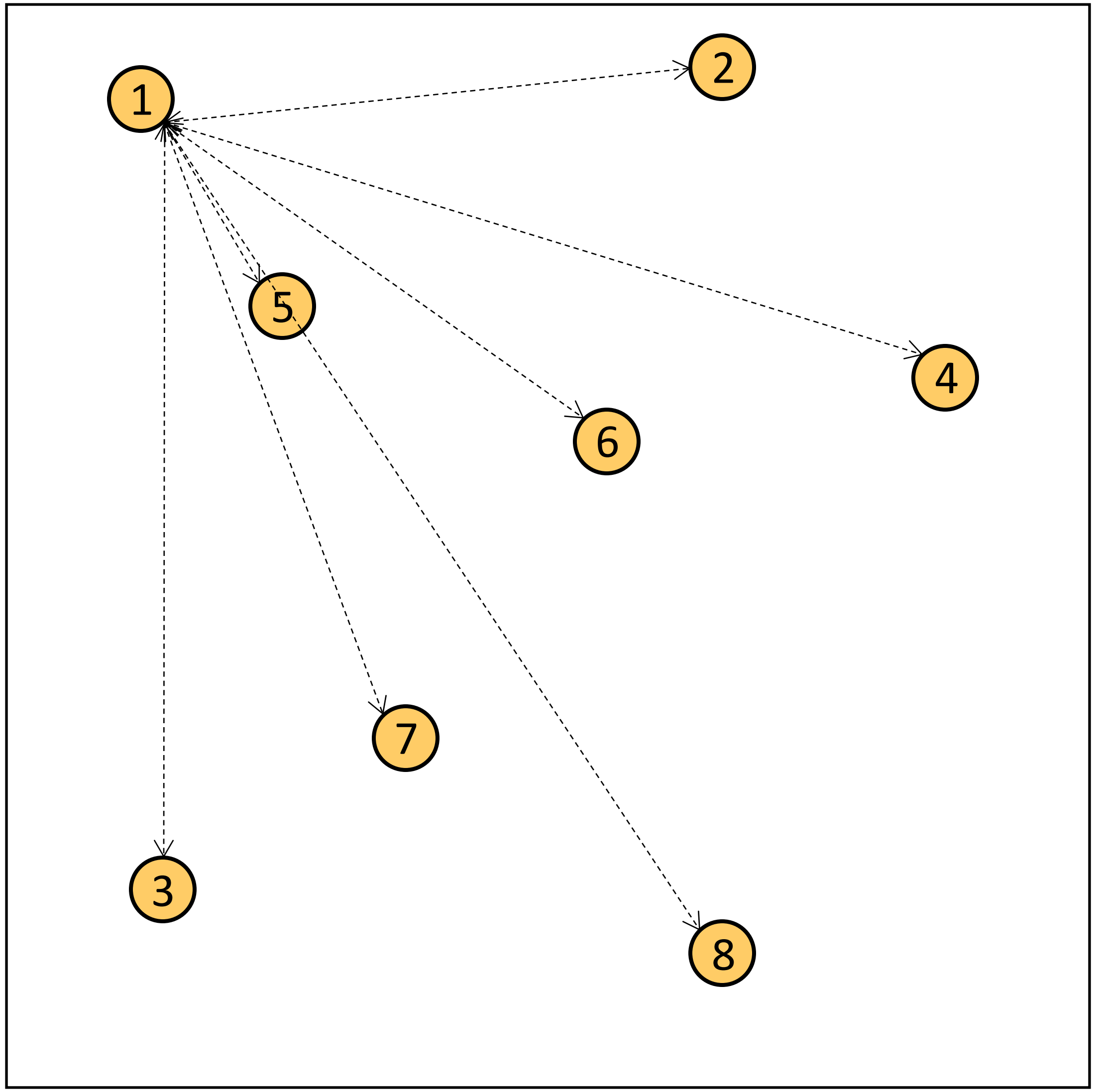
Return: Pair of points with smallest distance apart

Algorithm: Test every pair of points, return the closest

$f(n) \in \Theta(n^{\log_b a})$

Running Time: $O(n^2)$

Goal: $O(n \log n)$ $\approx T(n) = 2T(\frac{n}{2}) + \overset{f(n)}{n}$

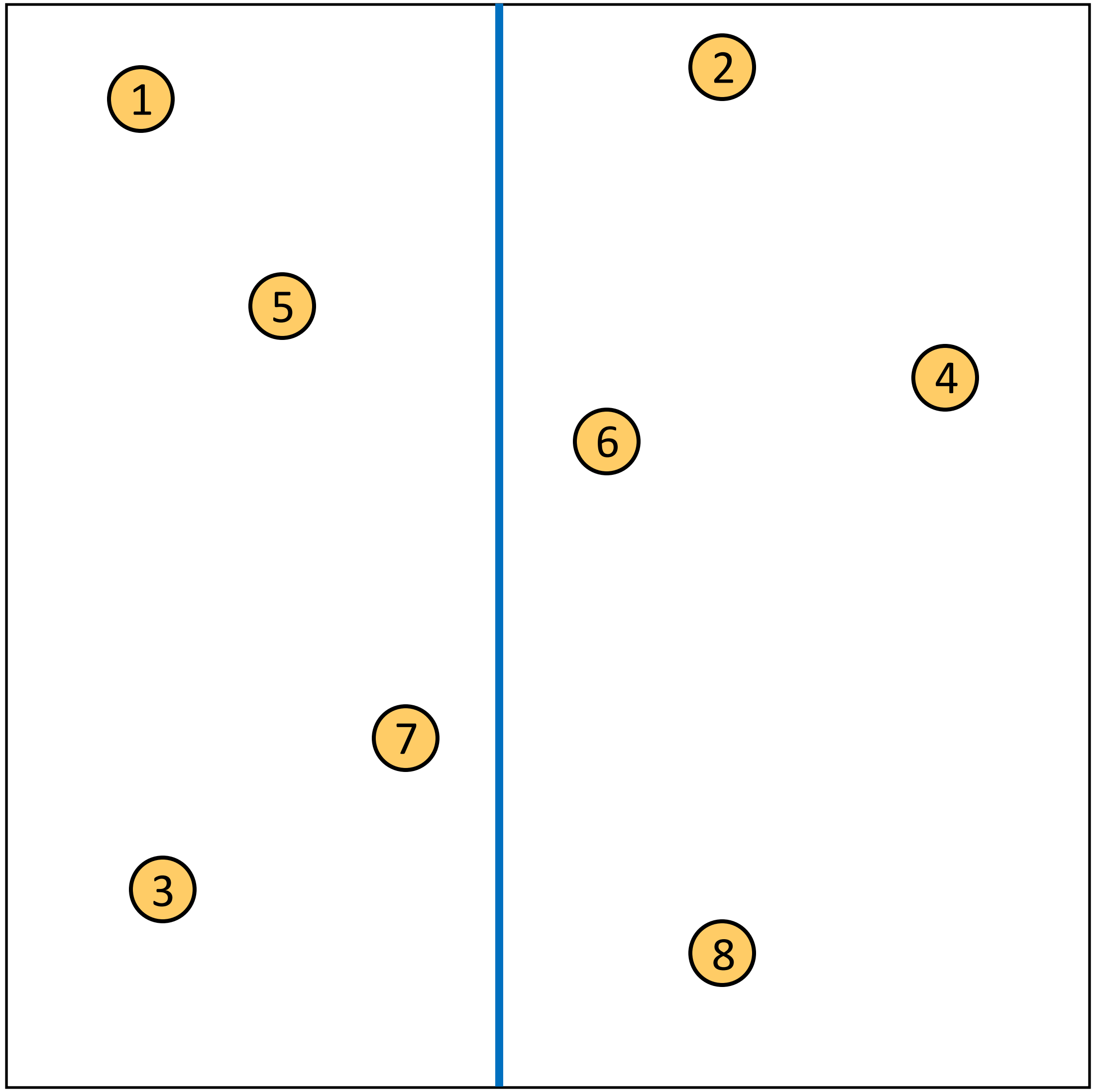


Closest Pair of Points: Divide and Conquer

need: sorted by x

Divide: How?

At median x coordinate



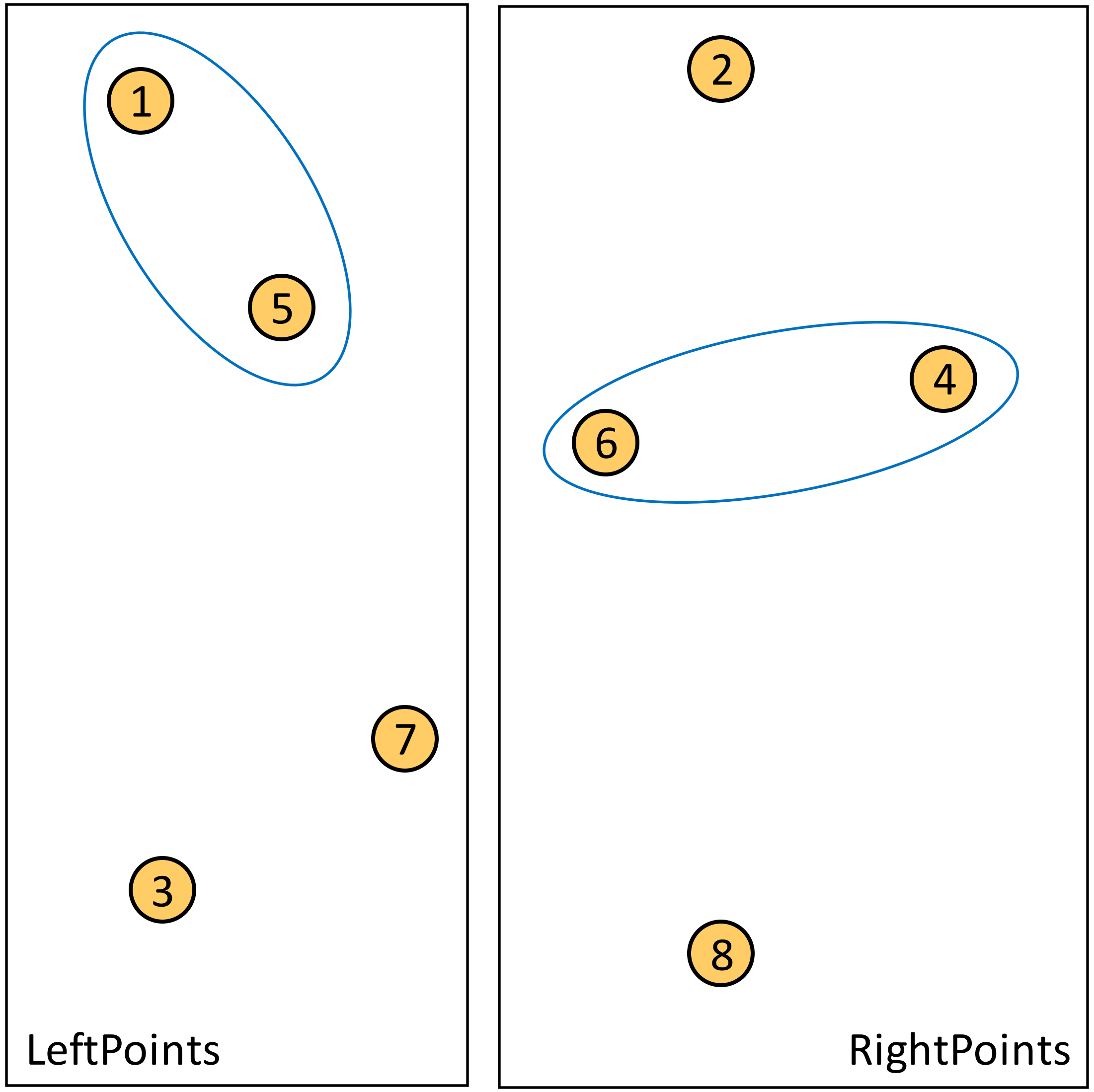
Closest Pair of Points: Divide and Conquer

Divide:

At median x coordinate

Conquer:

Recursively find closest pairs from LeftPoints and RightPoints



Closest Pair of Points: Divide and Conquer

Divide:

At median x coordinate

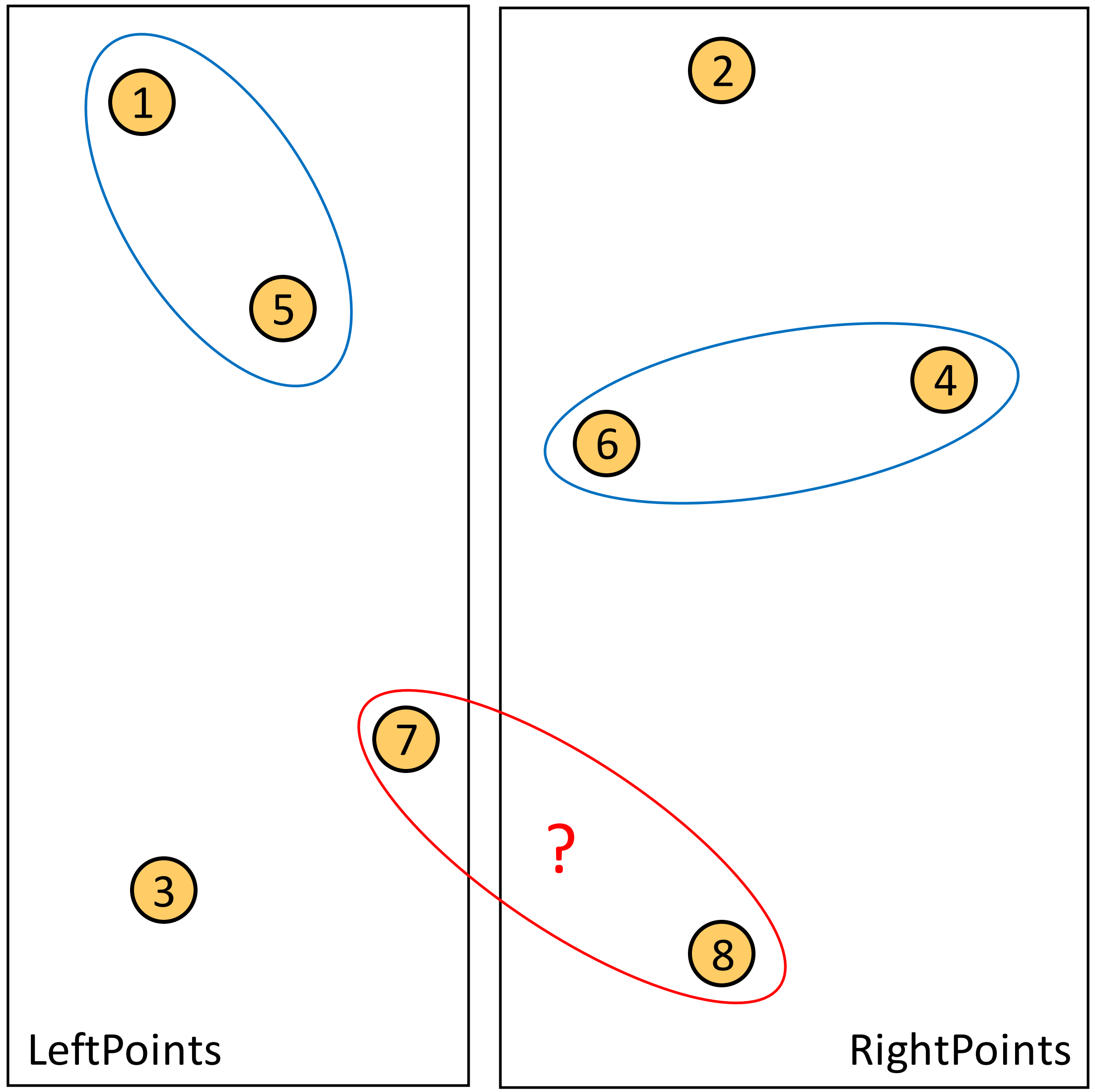
Conquer:

Recursively find closest pairs from LeftPoints and RightPoints

Combine:

Return smaller of left and right pairs

Problem?



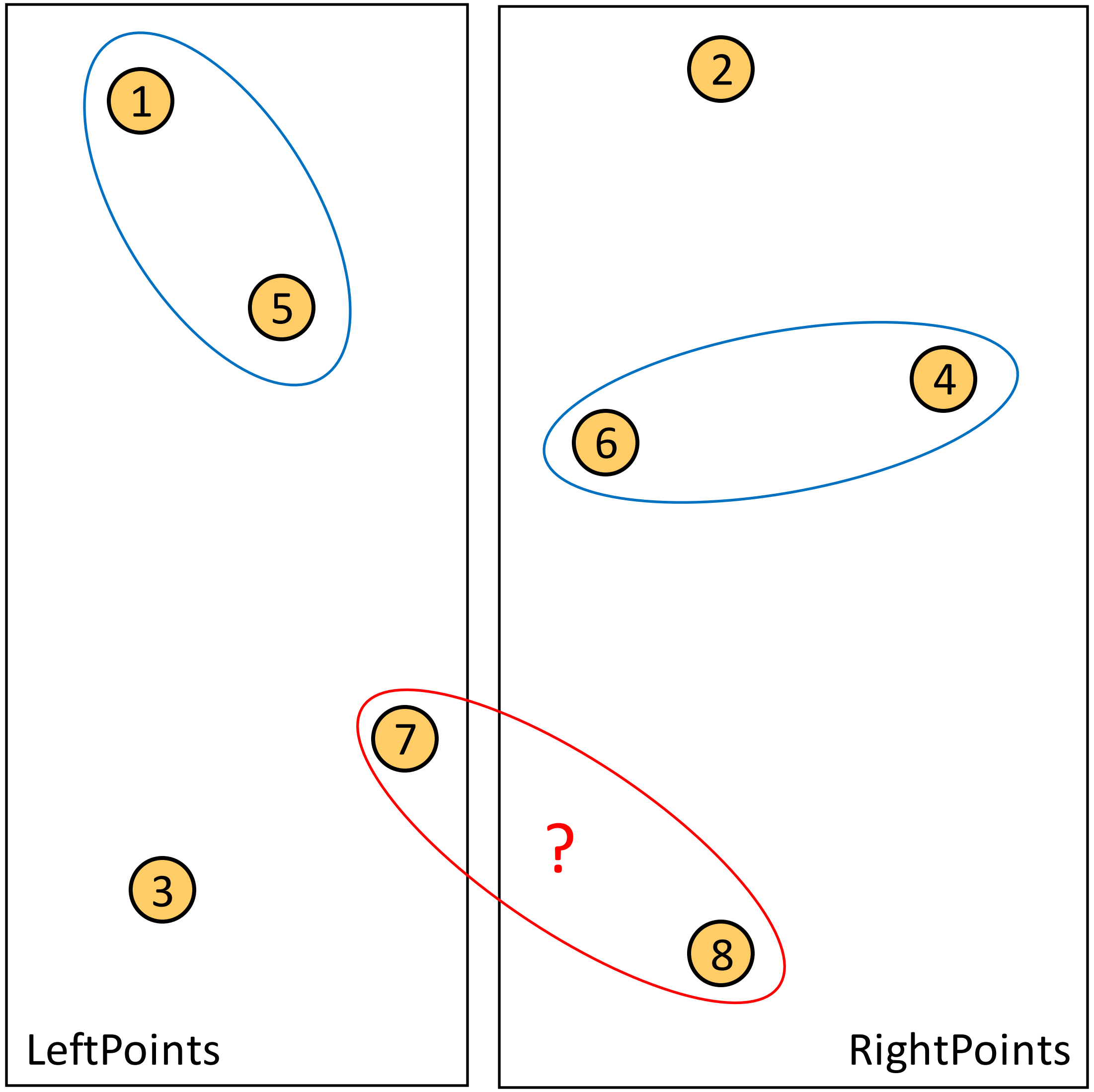
Closest Pair of Points: Divide and Conquer

Combine:

Case 1: Closest pair is completely in LeftPoints or RightPoints

Case 2: Closest pair spans our "cut"

Need to test points across the cut

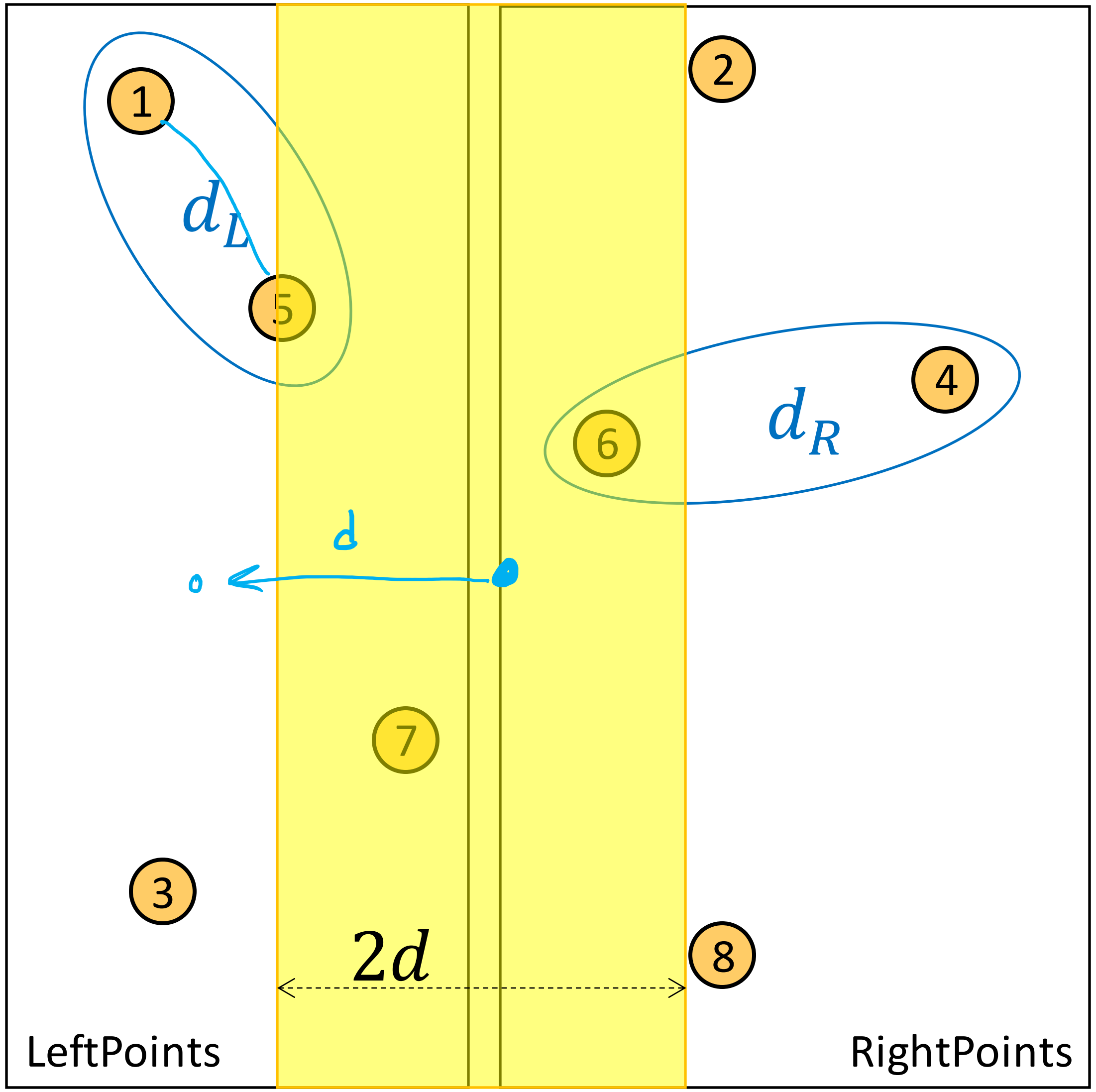


Spanning the Cut

Case 2: Closest pair spans our "cut"

Need to test points across the cut

- Compare all pairs of points within $d = \min\{d_L, d_R\}$ of the cut
- How many are there?



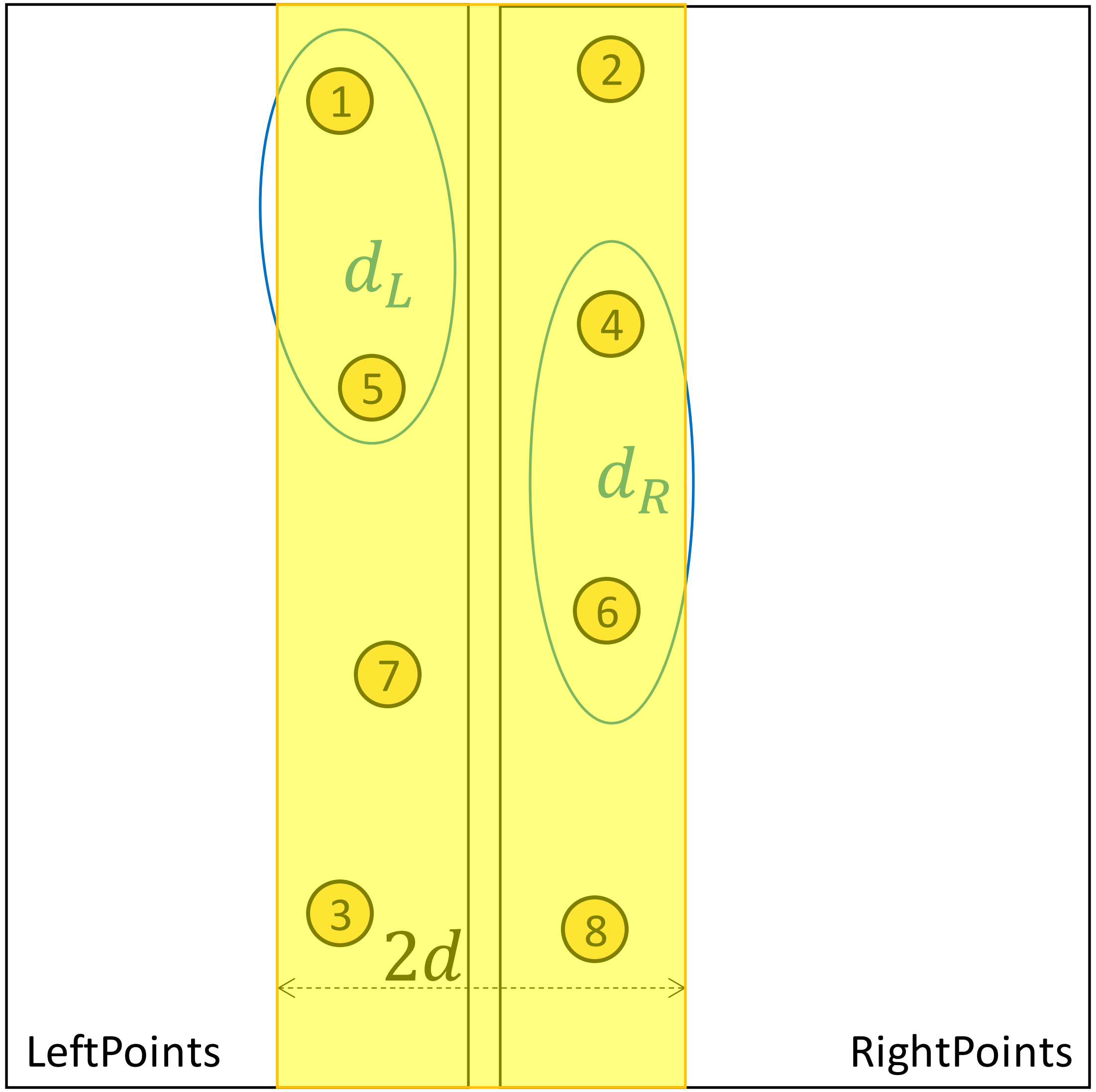
Spanning the Cut

Case 2: Closest pair spans our "cut"

Need to test points across the cut

- Compare all pairs of points within $d = \min\{d_L, d_R\}$ of the cut
- How many are there?
  - In the worst case, **all** of the points!

$$T(n) = 2T\left(\frac{n}{2}\right) + \Omega(n^2) \in \Omega(n^2)$$



Spanning the Cut

Case 2: Closest pair spans our "cut"

Need to test points across the cut

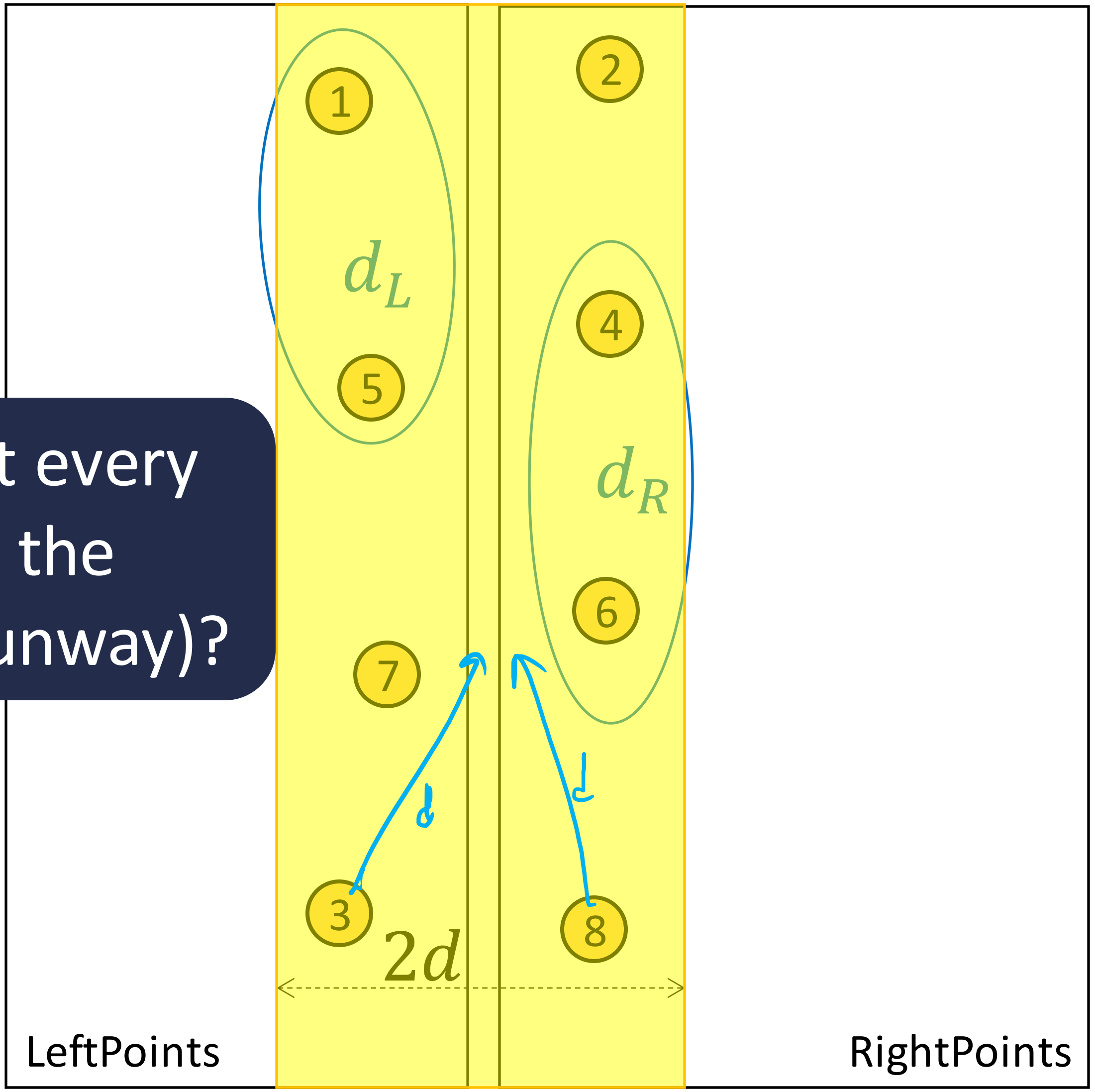
Compare all p
$d = \min\{d_L,$

How many ar

Do we need to test every pair of points in the boundary region (runway)?

In the worst case, **all** of the points!

$$T(n) = 2T\left(\frac{n}{2}\right) + \Omega(n^2) \in \Omega(n^2)$$



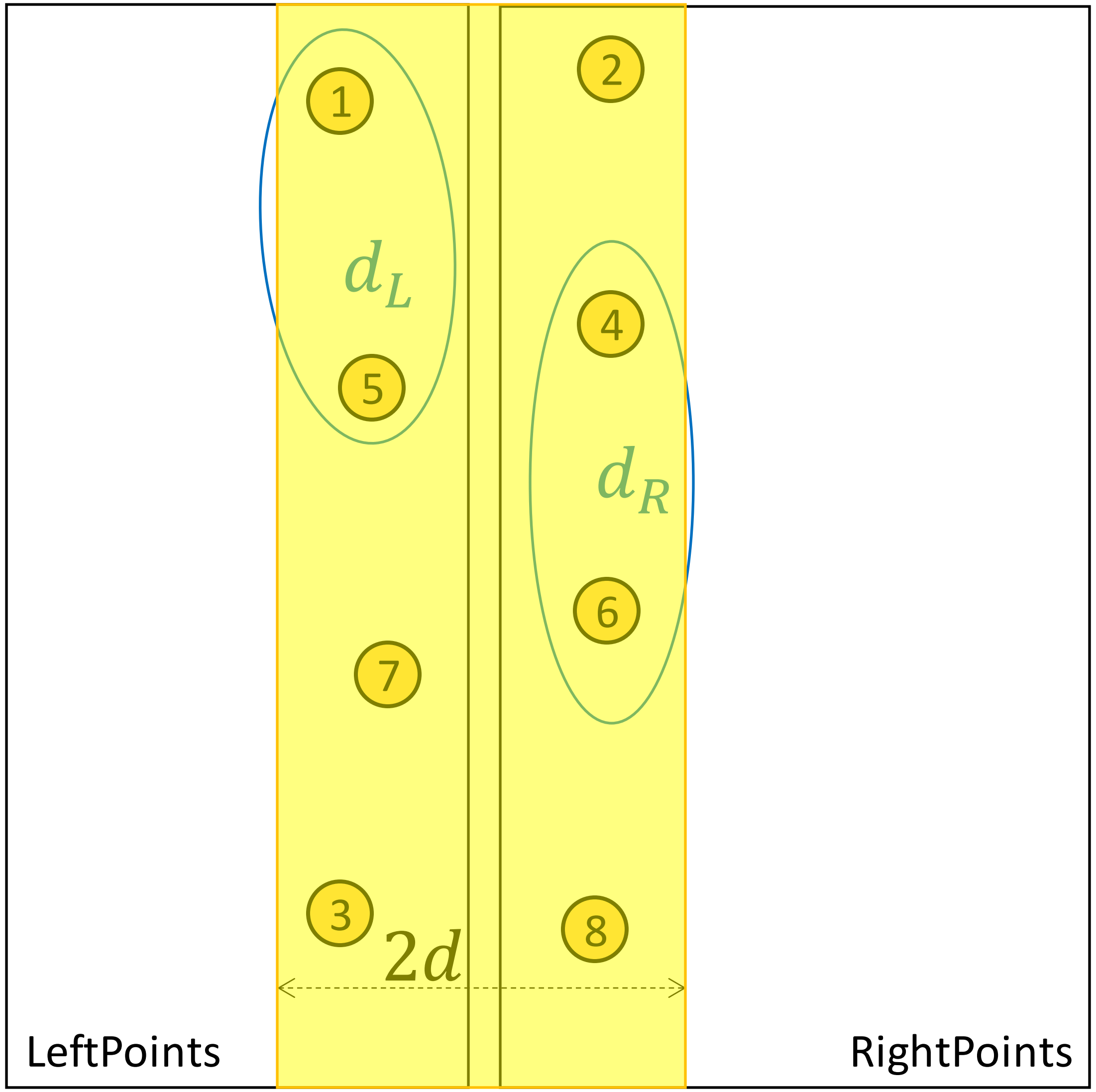
Spanning the Cut

Case 2: Closest pair spans our "cut"

Need to test points across the cut

Observation: We don't need to test all pairs!

Only need to test points within distance d of each another



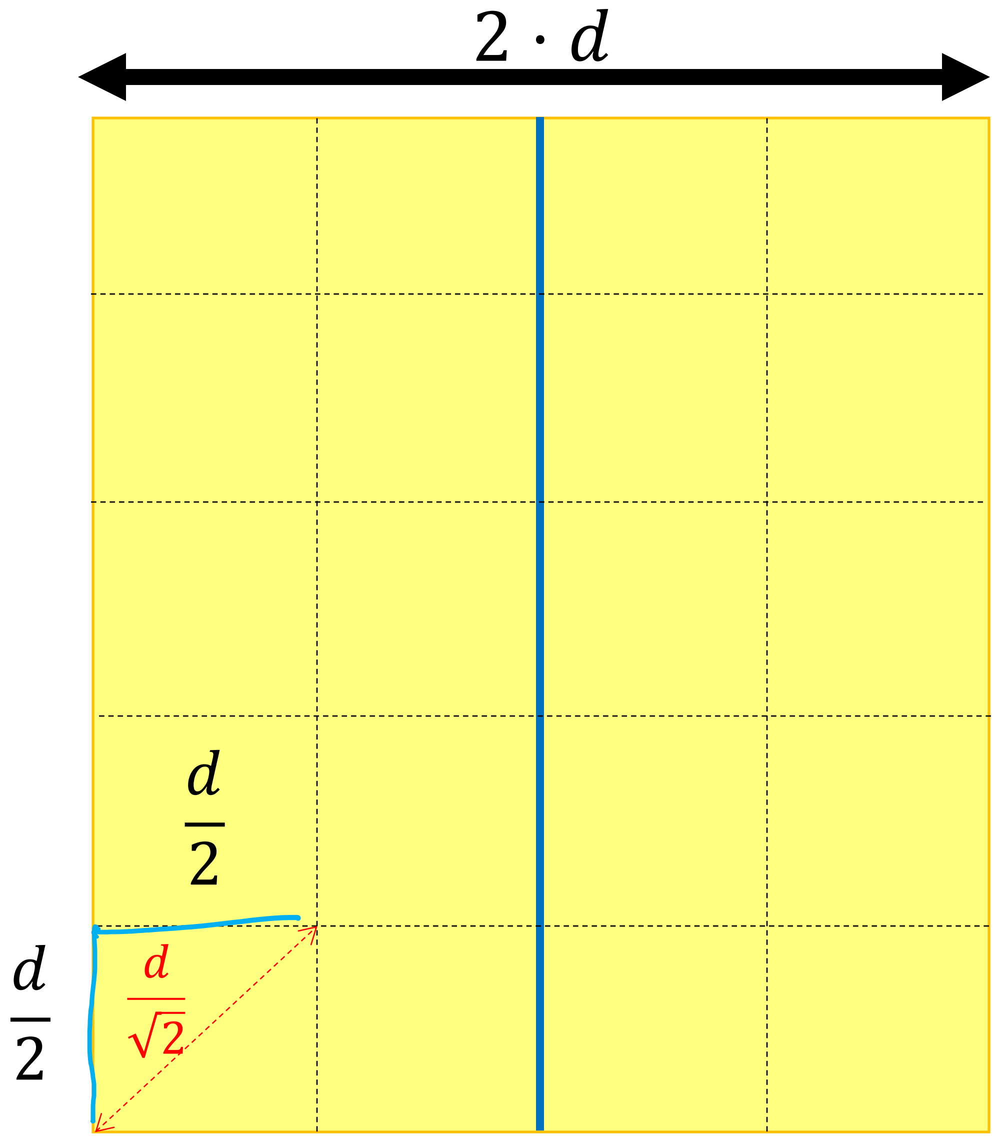
Reducing Search Space

Case 2: Closest pair spans our "cut"

Need to test points across the cut

Divide the runway into squares with dimension $d/2$

How many points can be in a square? **at most 1**



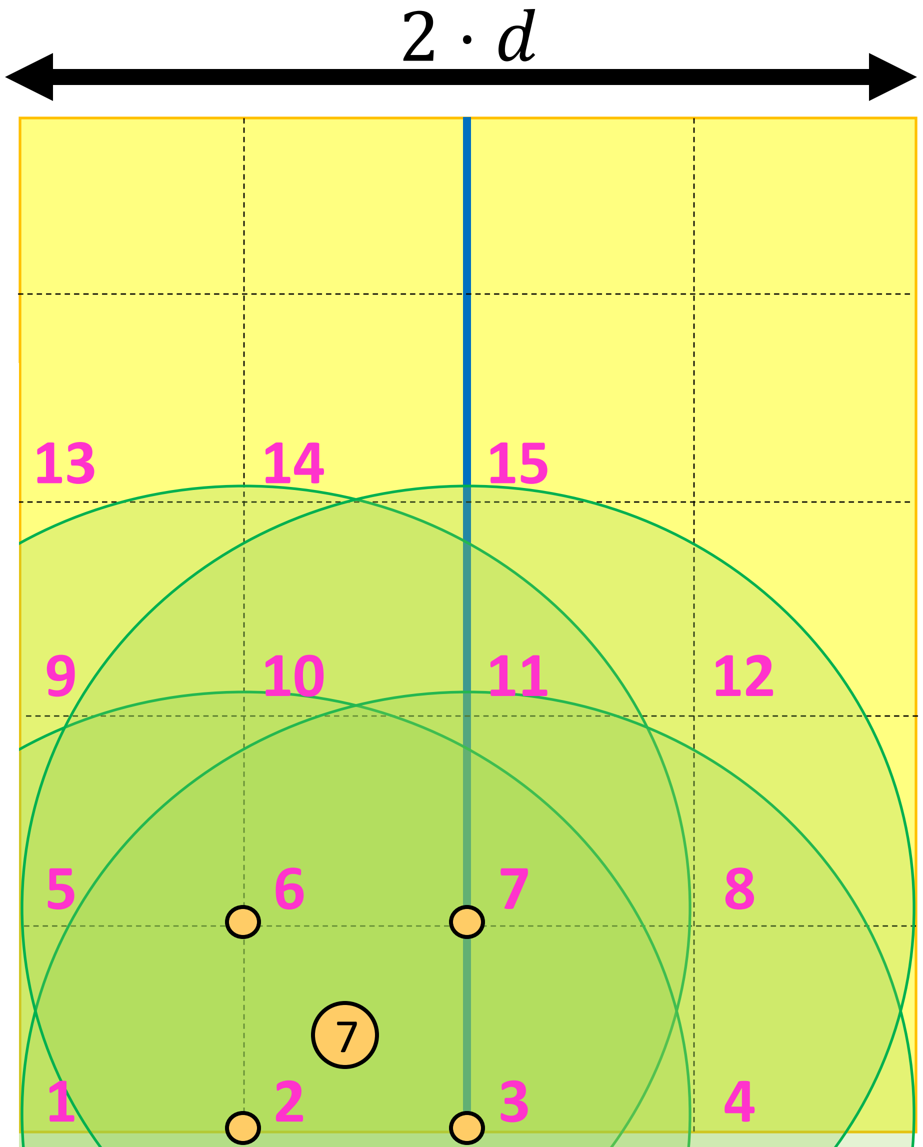
Reducing Search Space

Case 2: Closest pair spans our "cut"

Need to test points across the cut

Divide the runway into squares with dimension $d/2$

How many squares can contain a point $< d$ away?



at most **15**

Closest Pair of Points: Divide and Conquer

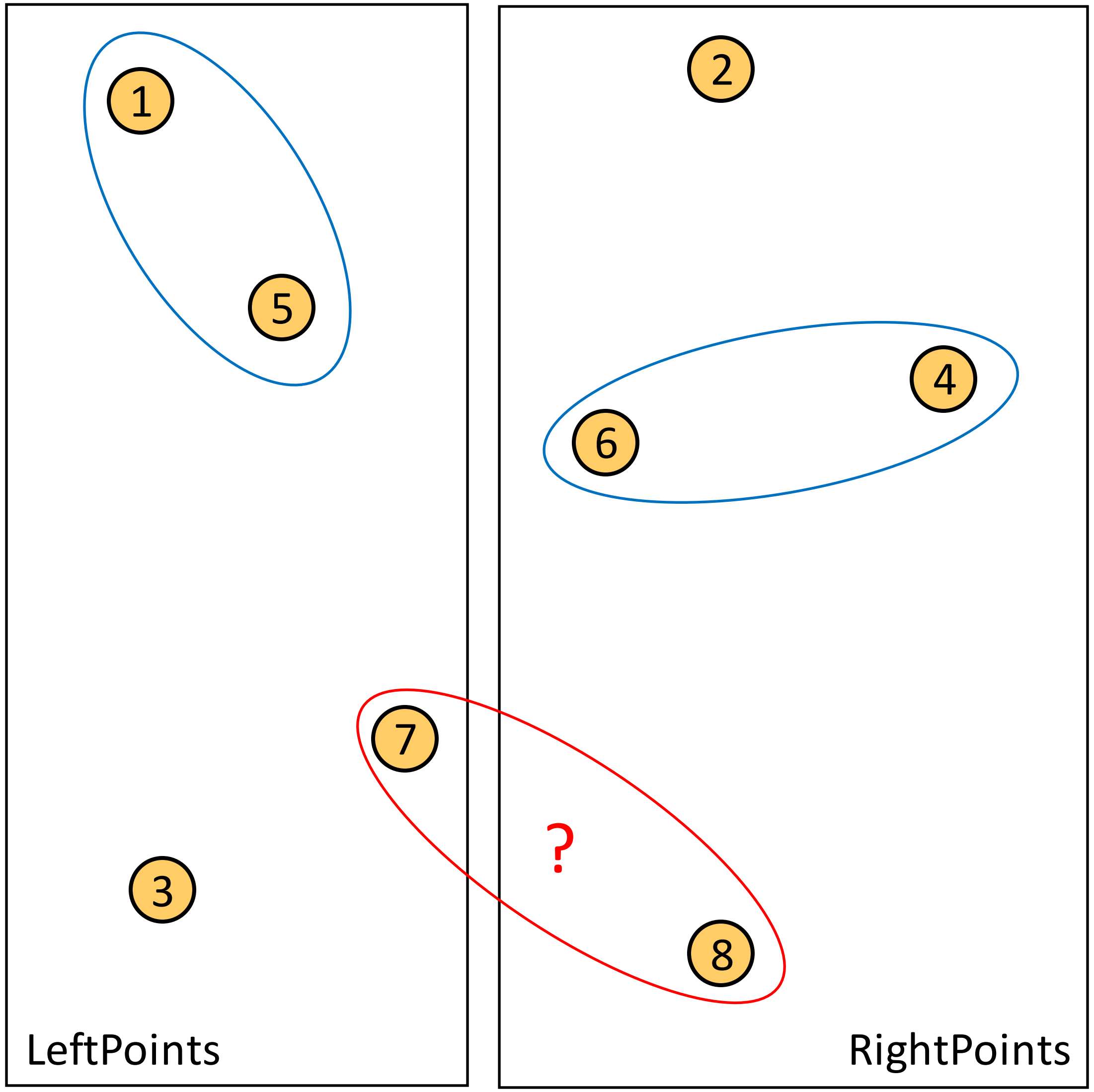
Initialization: Sort points by x-coordinate

Divide: Partition points into two lists of points based on x-coordinate

Conquer: Recursively compute the closest pair of points in each list

Combine:

- Construct list of points in the boundary
- Sort runway points by y-coordinate
- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points



Closest Pair of Points: Divide and Conquer

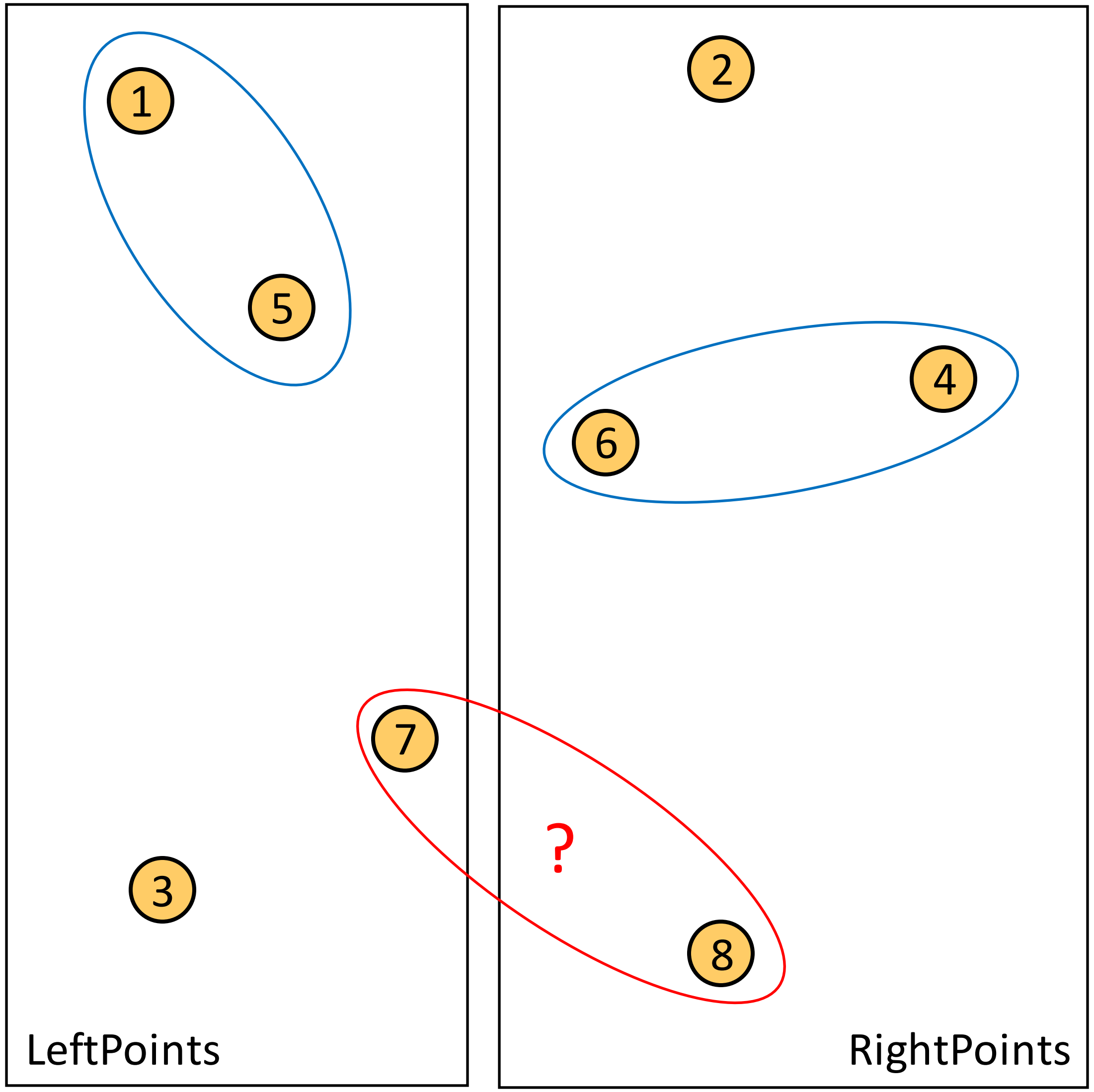
Initialization: Sort points by x-coordinate

Divide: Partition points into two lists of points

Looks like another $O(n \log n)$ algorithm – combine step is still too expensive

Combine:

- Construct list of points in t boundary
- Sort runway points by y-coordinate
- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points



Closest Pair of Points: Divide and Conquer

Initialization: Sort points by x-coordinate

Divide: Partition points into two lists of points based on x-coordinate

Conquer: Recursively compute the closest pair of points in each list

Combine:

- Construct list of points in the boundary
- Sort runway points by y-coordinate
- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points

Solution: Maintain additional information in the recursion

- Minimum distance among pairs of points in the list
- List of points sorted according to y-coordinate

Sorting runway points by y-coordinate now becomes a **merge**

Listing Points in the Boundary

LeftPoints:

Closest Pair: $(1, 5)$, $d_{1,5}$

Sorted Points: $[3,7,5,1]$

RightPoints:

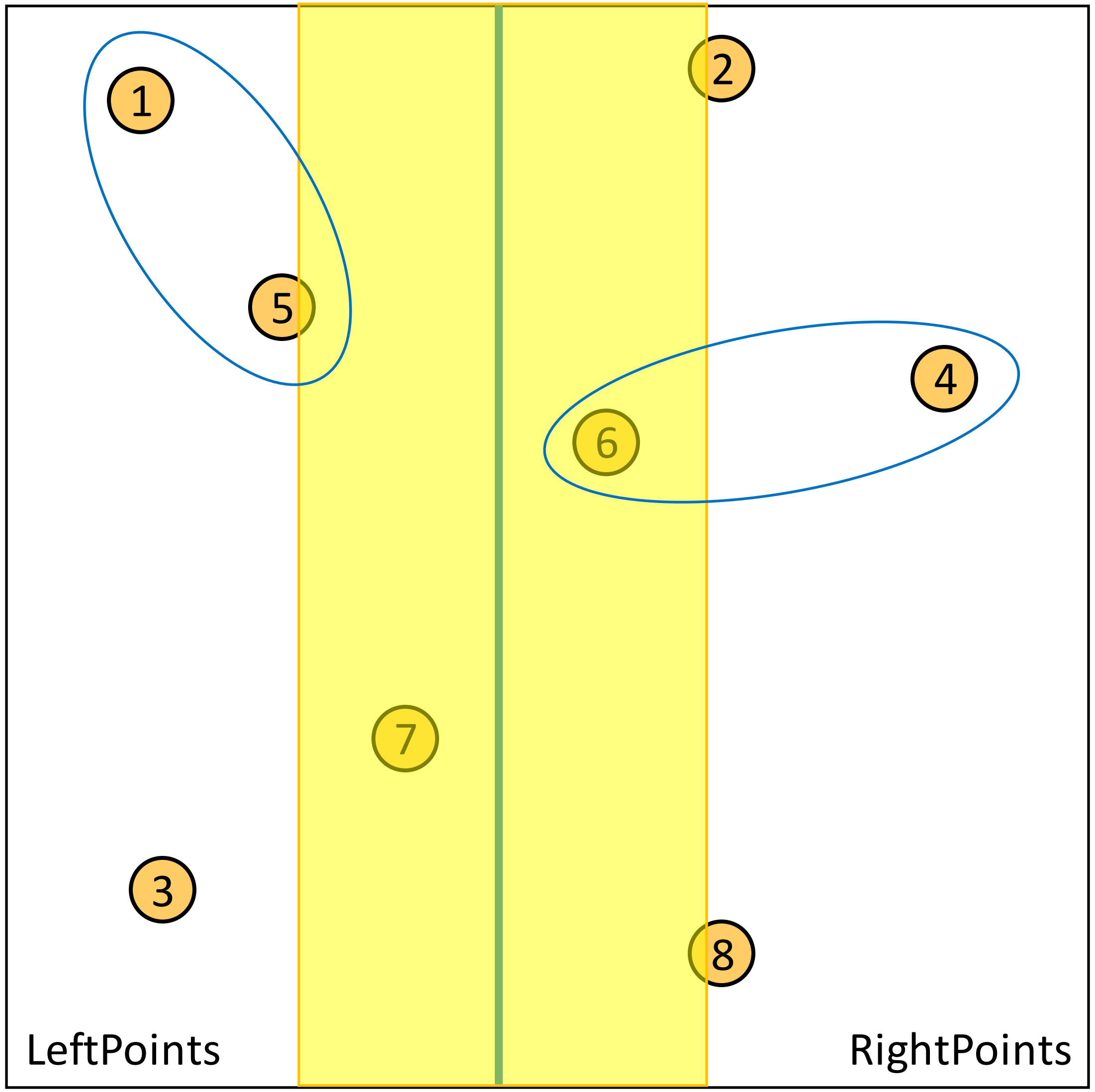
Closest Pair: $(4,6)$, $d_{4,6}$

Sorted Points: $[8,6,4,2]$

Merged Points: $[8,3,7,6,4,5,1,2]$

Runway Points: $[8,7,6,5,2]$

Both of these lists can be computed by a *single* pass over the lists



Closest Pair of Points: Divide and Conquer

Initialization: Sort points by x-coordinate

Divide: Partition points into two lists of points based on x-coordinate

Conquer: Recursively compute the closest pair of points in each list

Combine:

- Construct list of points in the boundary
- Sort runway points by y-coordinate
- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points

Initialization: Sort points by x-coordinate

Divide: Partition points into two lists of points based on x-coordinate

Conquer: Recursively compute the closest pair of points in each list

Combine:

- Merge sorted list of points by y-coordinate and construct list of points in the runway (sorted by y-coordinate)
- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points

Closest Pair of Points: Divide and Conquer

What is the running time?

$\Theta(n \log n)$

$$T(n) \begin{cases} \Theta(1) \\ 2T(n/2) \\ \Theta(n) \\ \Theta(n) \\ \Theta(1) \end{cases}$$

$\Theta(n \log n)$ **Initialization:** Sort points by x-coordinate

$\Theta(1)$ **Divide:** Partition points into two lists of points based on x-coordinate

$2T(n/2)$ **Conquer:** Recursively compute the closest pair of points in each list

Combine:

- $\Theta(n)$ Merge sorted list of points by y-coordinate and construct list of points in the runway (sorted by y-coordinate)
- $\Theta(n)$ Compare each point in runway to 15 points above it and save the closest pair
- $\Theta(1)$ Output closest pair among left, right, and runway points

$$T(n) = 2T(n/2) + \Theta(n)$$

Case 2 of Master's Theorem:

$$T(n) = \Theta(n \log n)$$