CS 3100
Data Structures and Algorithms 2
Lecture 8: Divide and Conquer

Co-instructors: Robbie Hott and Ray Pettit

Spring 2024

Readings in CLRS 4th edition:

- Section 4.1-4.4

Announcements

- PS3 due tomorrow
- PA2 coming soon
- Office hours
  - Prof Hott Office Hours: Mondays 11a-12p, Fridays 10-11a and 2-3p
  - Prof Pettit Office Hours: Mondays and Wednesdays 2:30-4:00p
  - TA office hours posted on our website
- Quizzes 1-2 coming February 29, 2024
  - Both quizzes taken the same day
  - If you have SDAC, please schedule for 1 exam (*not a quiz*)

Divide and Conquer

[CLRS Chapter 4]

Divide:

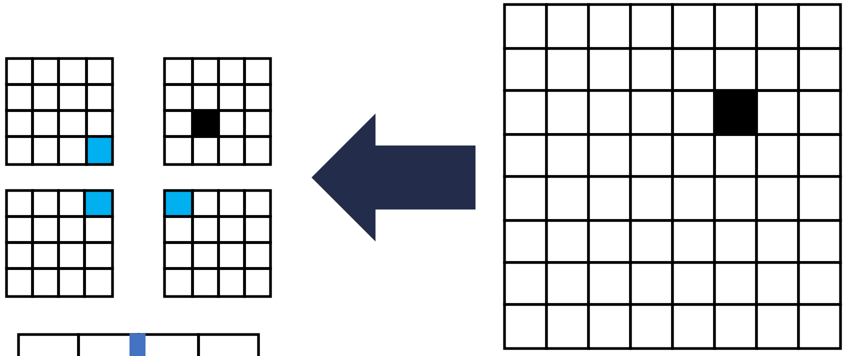
- Break the problem into multiple subproblems, each smaller instances of the original

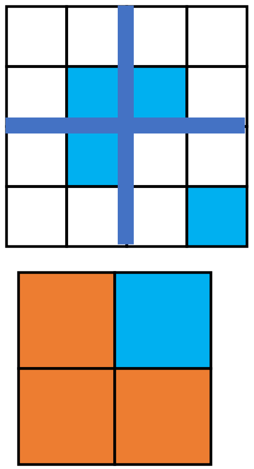
Conquer:

- If the suproblems are "large":
  - Solve each subproblem recursively
- If the subproblems are "small":
  - Solve them directly (base case)

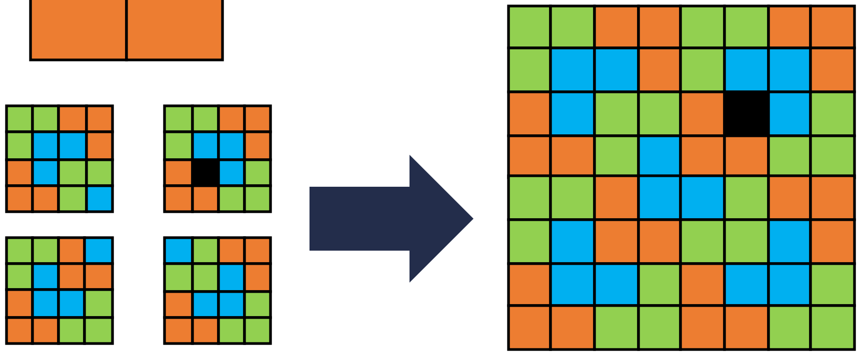
Combine:

- Merge solutions to subproblems to obtain solution for original problem





When is this an effective strategy?



Multiplication

Want to multiply large numbers together

$$\begin{array}{r} 4\,1\,0\,2 \\ \times\ 1\,8\,1\,9 \\ \hline \end{array}$$

n-digit numbers

How do we measure input size?

number of digits

What do we "count" for run time?

number of elementary operations
(single-digit multiplications)

"Schoolbook" Multiplication

How many multiplications?

```
      4 1 0 2
    × 1 8 1 9
   ----------
    3 6 9 1 8
    4 1 0 2
  3 2 8 1 6
+ 4 1 0 2
-------------
  7 4 6 1 5 3 8
```

n-digit numbers

n mults

n mults

n mults

n mults

n levels $\Rightarrow \Theta(n^2)$

What about cost of additions?

$\Theta(n^2)$

"Schoolbook" Multiplication

Can we do better?

How many multiplications?

$$
\begin{array}{r}
4102 \\
\times\ 1819 \\
\hline
36918 \\
4102 \\
32816 \\
+\ 4102 \\
\hline
7461538
\end{array}
$$

n-digit numbers

n mults

n mults

n mults

n mults

n levels $\Rightarrow \Theta(n^2)$

What about cost of additions?

$\Theta(n^2)$

Divide and Conquer Multiplication

1. Break into smaller subprojects

$$\begin{array}{r} a\;b \\ \times\; c\;d \\ \hline \end{array}$$

$$ab = 10^{\frac{n}{2}} a + b$$

$$cd = 10^{\frac{n}{2}} c + d$$

$$= 10^{n}(a \times c) +$$

$$10^{\frac{n}{2}}(a \times d + b \times c) +$$

$$(b \times d)$$

Divide and Conquer Multiplication

Divide:

- Break n-digit numbers into four numbers of $n/2$ digits each (call them a, b, c, d)

Conquer:

- If $n > 1$:
  - Recursively compute ac, ad, bc, bd
- If $n = 1$: (i.e. one digit each)
  - Compute ac, ad, bc, bd directly (base case)

Combine:

- $10^n(ac) + 10^{n/2}(ad + bc) + bd$

For simplicity, assume that $n = 2^k$ is a power of 2

Divide and Conquer Multiplication

2. Use recurrence relation to express recursive running time

$$10^n(ac) + 10^{n/2}(ad + bc) + bd$$

Recursively solve

$$T(n)$$

Divide and Conquer Multiplication

2. Use recurrence relation to express recursive running time

$$10^n(ac) + 10^{n/2}(ad + bc) + bd$$

Recursively solve

$$T(n) = 4T\left(\frac{n}{2}\right)$$

Need to compute 4 multiplications, each of size $n/2$

Divide and Conquer Multiplication

2. Use recurrence relation to express recursive running time

$$10^n(ac) + 10^{n/2}(ad + bc) + bd$$

Recursively solve

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

Need to compute 4 multiplications, each of size $n/2$

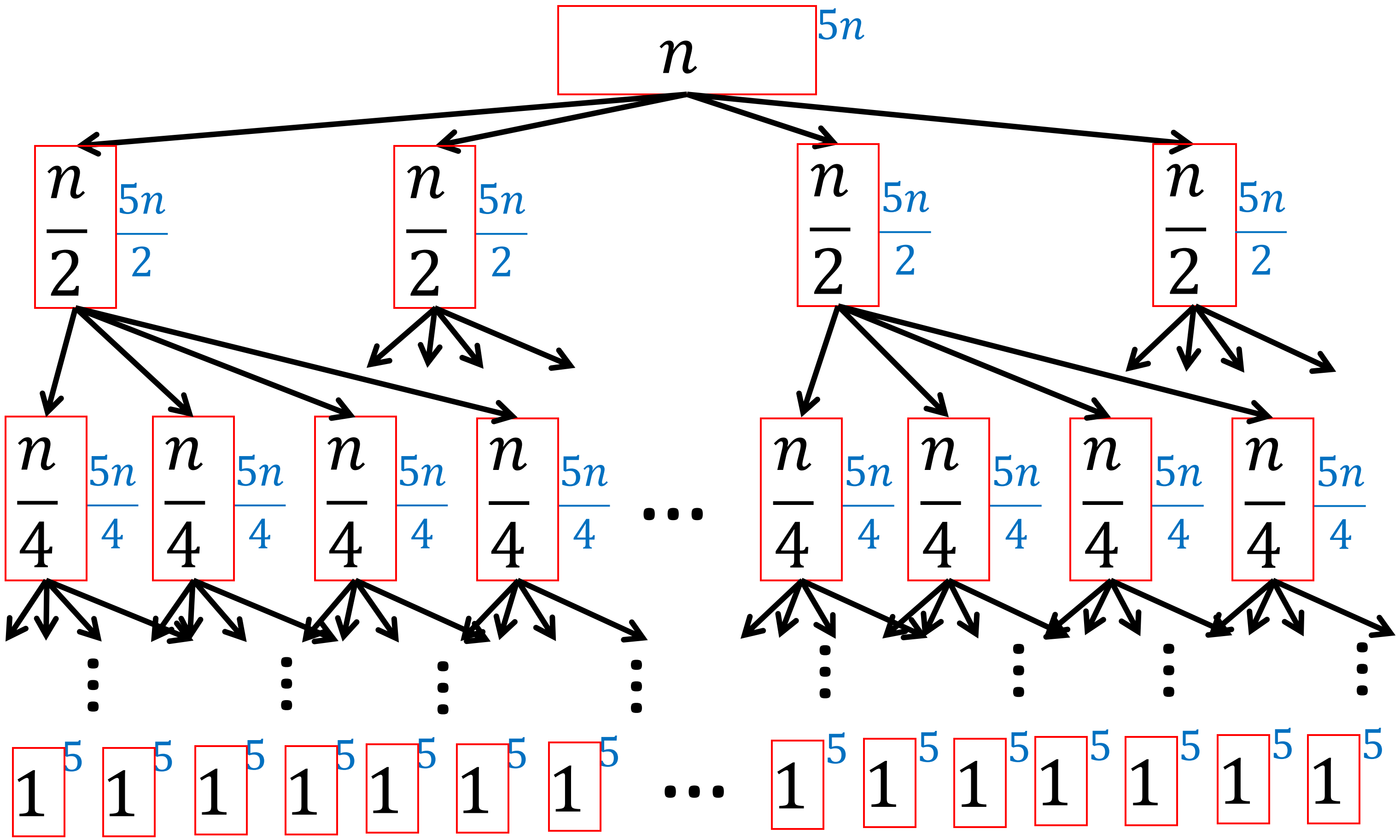
2 shifts and 3 additions on n-bit values

Divide and Conquer Multiplication

3. Use asymptotic notation to simplify

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

$$T(n) = 5n \sum_{i=0}^{\log_2 n} 2^i$$



$5n$

$\frac{4}{2} \cdot 5n$

$\frac{16}{4} \cdot 5n$

⋮

$2^{\log_2 n} \cdot 5n$

Divide and Conquer Multiplication

3. Use asymptotic notation to simplify

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

$$T(n) = 5n \sum_{i=0}^{\log_2 n} 2^i$$

$$T(n) = 5n \frac{2^{\log_2 n+1} - 1}{2 - 1}$$

$$T(n) = 5n(2n - 1) \;\; = \Theta(n^2)$$

No better than the schoolbook method!

Karatsuba Multiplication

1. Break into smaller subproblems

$$
\begin{array}{r}
a\ b \\
\times\ c\ d \\
\hline
\end{array}
\qquad
\begin{array}{l}
= 10^{\frac{n}{2}} a + b \\
= 10^{\frac{n}{2}} c + d
\end{array}
$$

$$= \quad 10^{n}(a \times c) +$$

$$10^{\frac{n}{2}}(a \times d + b \times c) +$$

$$(b \times d)$$

Recall: previous divide-and-conquer recursively computed ac, ad, bc, bd

Karatsuba Multiplication

$$10^n(ac) + 10^{\frac{n}{2}}(ad + bc) + bd$$

	a	b
×	c	d

Can't avoid these

This can be simplified!

$$(a + b)(c + d) =$$

$$ac + ad + bc + bd$$

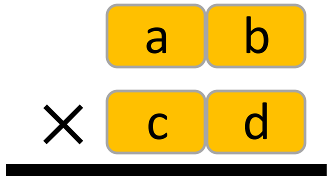
$$ad + bc = (a + b)(c + d) - ac - bd$$

Two multiplications

One multiplication

Karatsuba Multiplication

2. Use recurrence relation to express recursive running time



$$10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$$

Recursively solve

$$T(n) =$$

Karatsuba Multiplication

2. Use recurrence relation to express recursive running time

	a	b
×	c	d

$$10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$$

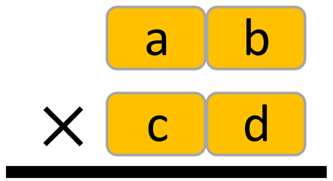
Recursively solve

$$T(n) = 3T\left(\frac{n}{2}\right)$$

Need to compute **3** multiplications, each of size $n/2$: ac, bd, $(a+b)(b+c)$

Karatsuba Multiplication

2. Use recurrence relation to express recursive running time



$$10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$$

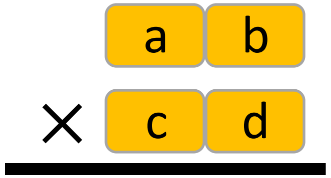
Recursively solve

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Need to compute **3** multiplications, each of size $n/2$: ac, bd, $(a+b)(b+c)$

2 shifts and 6 additions on n-bit values

Karatsuba Multiplication



Divide:

- Break n-digit numbers into four numbers of $^n/_2$ digits each (call them a, b, c, d)

Conquer:

- If $n > 1$:
  - Recursively compute ac, bd, $(a+b)(c+d)$
- If $n = 1$:
  - Compute ac, bd, $(a+b)(c+d)$ directly (base case)

Combine:

- $10^n(ac) + 10^{n/2}\big((a+b)(c+d) - ac - bd\big) + bd$

Karatsuba Multiplication

a b
× c d

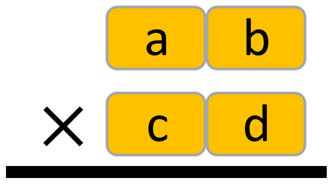
1. Recursively compute: $ac, bd, (a+b)(c+d)$
2. $(ad+bc) = (a+b)(c+d) - ac - bd$
3. Return $10^n(ac) + 10^{\frac{n}{2}}(ad+bc) + bd$

Pseudocode:

1. $x \leftarrow$ Karatsuba(a, c)
2. $y \leftarrow$ Karatsuba(a, d)
3. $z \leftarrow$ Karatsuba$(a+b, c+d) - x - y$
4. Return $10^n x + 10^{n/2} z + y$

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Karatsuba Multiplication



1. Recursively compute: $ac, bd, (a+b)(c+d)$
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<u>Pseudocode:</u>

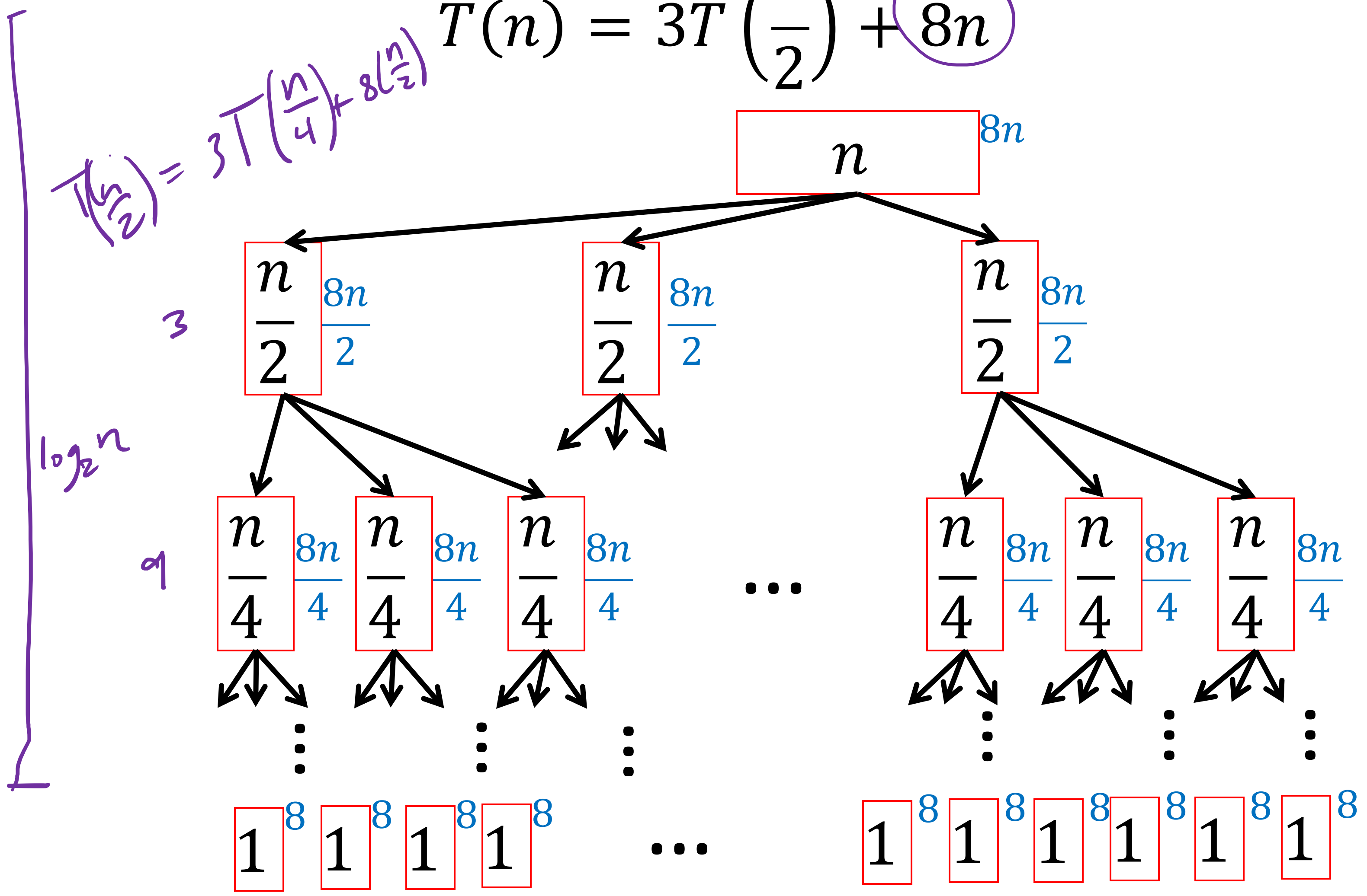
1. $x \leftarrow$ Karatsuba(a, c)
2. $y \leftarrow$ Karatsuba(a, d)
3. $z \leftarrow$ Karatsuba$(a+b, c+d) - x - y$
4. Return $10^n x + 10^{n/2} z + y$

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Karatsuba

3. Use asymptotic notation to simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$



$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}/_{2}\right)^i$$

$8n \cdot 1$

$8n \cdot \frac{3}{2}$

$8n \cdot \frac{9}{4}$

$8n \cdot \frac{3^{\log_2 n}}{2^{\log_2 n}}$

Karatsuba

3. Use asymptotic notation to simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}/_{2}\right)^{i}$$

$$T(n) = 8n \frac{\left({}^{3}/_{2}\right)^{\log_2 n+1} - 1}{{}^{3}/_{2} - 1}$$

Math, math, and more math…(on board, see lecture supplement)

Karatsuba

$3 = 2^{\log_2 3}$

$$T(n) = 8n \frac{(3/2)^{\log_2 n + 1} - 1}{3/2 - 1} = 16n\left(\left(\frac{3}{2}\right)^{\log_2 n + 1} - 1\right)$$

$\frac{1}{2}$

$\frac{1}{2} = 2^{-1}$

$$= 16n\left(\left(\frac{2^{\log_2 3}}{2}\right)^{\log_2 n + 1} - 1\right)$$

$$= 16n\left(\left(2^{\log_2 3 - 1}\right)^{\log_2 n + 1} - 1\right)$$

$$= 16n\left(2^{\log_2 3 \log_2 n + \log_2 3 - \log_2 n - 1} - 1\right)$$

$$= 16n\left(2^{\log_2 3 \log_2 n} \cdot 2^{\log_2 3} \cdot 2^{-\log_2 n} \cdot 2^{-1} - 1\right) = 24n^{\log_2 3} - 16n$$

$n^{\log_2 3}$ 3 $\frac{1}{2^{\log_2 n}} = \frac{1}{n}$ $\frac{1}{2}$

negative

$\in \Theta(n^{\log_2 3}) \approx \Theta(n^{1.585})$

$2^{\log_2 3 \log_2 n}$

$(2^{\log_2 n})^{\log_2 3}$

$n^{\log_2 3}$

Karatsuba

Karatsuba Multiplication

$$T(n) = 8n\frac{\left({}^{3}/_{2}\right)^{\log_2 n+1}-1}{{}^{3}/_{2}-1}$$

How to simplify this (using asymptotic notation)?

Drop constant multiples

Karatsuba Multiplication

$$T(n) = 8n \frac{\left({}^{3}/_{2}\right)^{\log_2 n+1} - 1}{{}^{3}/_{2} - 1}$$

How to simplify this (using asymptotic notation)?

$$= \Theta\left(n\left(\left({}^{3}/_{2}\right)^{\log_2 n+1} - 1\right)\right)$$

Drop constant multiples

$$= \Theta\left(\frac{3}{2}n \cdot \left({}^{3}/_{2}\right)^{\log_2 n} - n\right)$$

Distribute terms

Karatsuba Multiplication

$$T(n) = 8n \frac{(3/2)^{\log_2 n+1} - 1}{3/2 - 1}$$

How to simplify this (using asymptotic notation)?

$$= \Theta\left(n\left((3/2)^{\log_2 n+1} - 1\right)\right)$$

Drop constant multiples

$$= \Theta\left(\frac{3}{2} n \cdot (3/2)^{\log_2 n} - n\right)$$

Distribute terms

$$= \Theta\left(n \cdot (3/2)^{\log_2 n}\right)$$

Drop constants and low-order terms

Karatsuba Multiplication

$$T(n) = \Theta\left(n \cdot \left({}^{3}/_{2}\right)^{\log_2 n}\right)$$

How to simplify this (using asymptotic notation)?

Properties of logarithms:

$$2^{\log_2 n} = n$$

$$3^{\log_2 n} = 2^{\log_2\left(3^{\log_2 n}\right)} = 2^{(\log_2 n)(\log_2 3)} = \left(2^{\log_2 n}\right)^{\log_2 3} = n^{\log_2 3}$$

$a^{bc} = \left(a^b\right)^c$

$2^{\log_2 n} = n$

$\log a^b = b \log a$

$2^{\log_2 n} = n$

Karatsuba Multiplication

$$T(n) = \Theta\left(n \cdot \left({}^{3}/_{2}\right)^{\log_2 n}\right)$$

$$= \Theta\left(n \cdot \left(\frac{3^{\log_2 n}}{2^{\log_2 n}}\right)\right)$$

$$= \Theta\left(n \cdot \left(\frac{n^{\log_2 3}}{n}\right)\right)$$

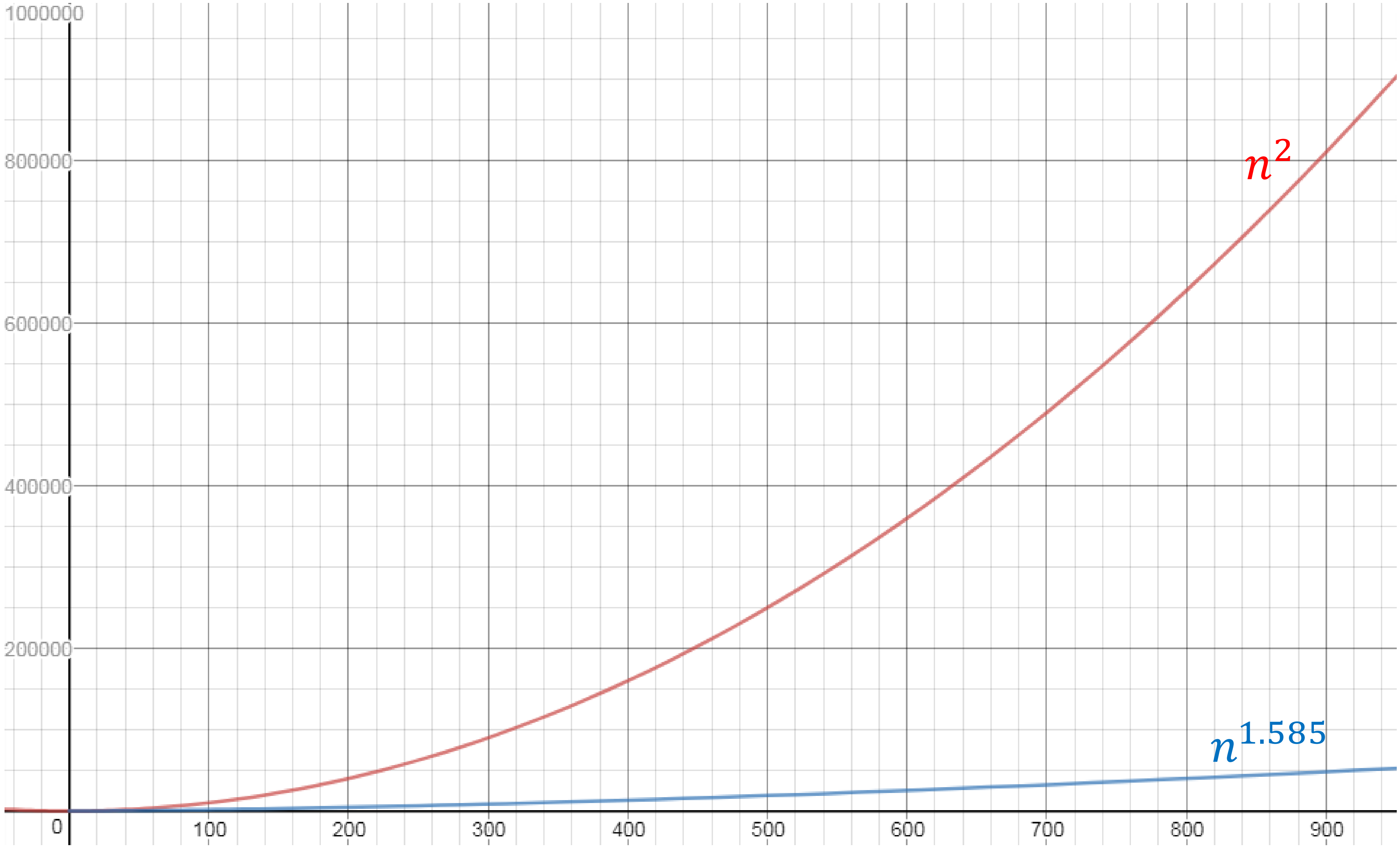
$$= \Theta\left(n^{\log_2 3}\right) \approx \Theta\left(n^{1.585}\right)$$

How to simplify this (using asymptotic notation)?

$$2^{\log_2 n} = n$$

$$3^{\log_2 n} = n^{\log_2 3}$$

Strictly better than schoolbook method!


1000000
800000
600000
400000
200000
0
100
200
300
400
500
600
700
800
900
n^2
$n^{1.585}$

Analyzing Divide and Conquer

1. Break into smaller subproblems
2. Use recurrence relation to express recursive running time
3. Use asymptotic notation to simplify

Divide: $D(n)$ time

Conquer: Recurse on smaller problems of size $s_1, \ldots, s_k$

Combine: $C(n)$ time

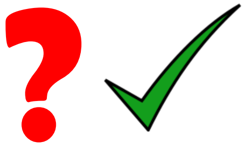
Recurrence:

- $T(n) = D(n) + \sum_{i \in [k]} T(s_i) + C(n)$

Recurrence Solving Techniques

Tree

get a picture of recursion



Guess/Check

guess and use induction to prove

“Cookbook”

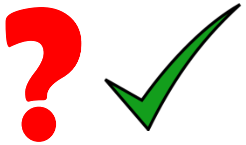
MAGIC!

Substitution

substitute in to simplify

Recurrence Solving Techniques

Tree



Guess/Check (induction)

"Cookbook"

Substitution

Guess and Check Blueprint

Show: $T(n) = O(g(n))$

Consider: $g_*(n) = c \cdot g(n)$ for some constant c

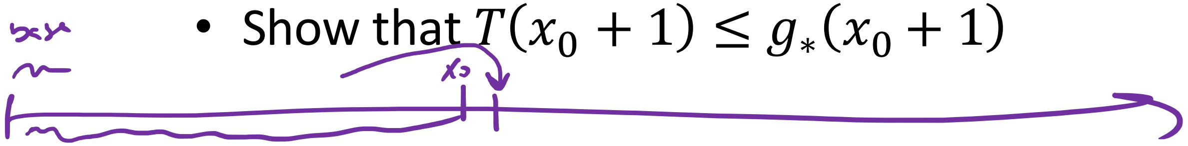
$g_*(n) \in O(g(n))$

Goal: show $\exists n_0$ such that $\forall n > n_0, T(n) \le g_*(n)$

- (definition of big-O)

Technique: Induction

- Base cases:
  - Show $T(1) \le g_*(1)$ (sometimes, may need to consider additional base cases)
- Hypothesis:
  - $\forall n \le x_0, T(n) \le g_*(n)$
- Inductive step:
  - Show that $T(x_0 + 1) \le g_*(x_0 + 1)$



Need to ensure that in inductive step, can either appeal to a base case or to the inductive hypothesis

Mergesort Guess and Check

$$T(n) = 2T(n/2) + n \quad \in \Theta(n \log n)$$

$g(n)$

$T(n) \in O(g_*(n))$ &

$\in O(g(n))$

$g_*(n) = n \log_2 n$

$\forall n \geq 1$ goal: $T(n) \leq n \log_2 n$ $T(1)$, $T(2)$

Hyp: $\forall n \leq x_0$ $T(n) \leq n \log_2 n$

$\leq x_0$

Inductive step: $T(x_0+1) = 2T\left(\frac{x_0+1}{2}\right) + (x_0+1)$

$\leq 2\left(\left(\frac{x_0+1}{2}\right)\log_2\left(\frac{x_0+1}{2}\right)\right) + x_0+1$

$= (x_0+1)(\log_2(x_0+1) - 1) + x_0 + 1$

$= (x_0+1)\log_2(x_0+1) - (x_0+1) + (x_0+1)$

$T(n) \in O(n \log n)$

Karatsuba Analysis using Guess and Check

$$T(n) = 3T(n/2) + 8n \quad \in O(n^{1.6})$$

Goal: $T(n) \leq 3000\, n^{1.6} = O(n^{1.6})$

Base case: $T(1) = 8 \leq 3000$

Hypothesis: $\forall n \leq x_0, \;\; T(n) \leq 3000n^{1.6}$

Inductive step: Show $T(x_0 + 1) \leq 3000(x_0 + 1)^{1.6}$

Karatsuba Guess and Check (Loose)

$$T(n) = 3T(n/2) + 8n$$

Hypothesis: $\forall n \leq x_0 \colon T(n) \leq 3000n^{1.6}$

Show: $T(x_0 + 1) \leq 3000(x_0 + 1)^{1.6}$

$\leq x_0$

$$T(x_0 + 1) = 3T\left(\frac{x_0 + 1}{2}\right) + 8(x_0 + 1)$$ Recurrence definition

$$\leq 3\left(3000\left(\frac{x_0 + 1}{2}\right)^{1.6}\right) + 8(x_0 + 1)$$ Inductive hypothesis

Karatsuba Guess and Check (Loose)

$T(x_0+1) \leq 3\left(3000\left(\frac{x_0+1}{2}\right)^{1.6}\right) + 8(x_0+1)$

$= \frac{3}{2^{1.6}}\left(3000(x_0+1)^{1.6}\right) + 8(x_0+1)$

$\leq .997 \cdot 3000(x_0+1)^{1.6} + 8(x_0+1)$

$= (1-.003)\, 3000(x_0+1)^{1.6} + 8(x_0+1)$

$= 3000(x_0+1)^{1.6} - 9(x_0+1)^{1.6} + 8(x_0+1)$

negative

$\leq 3000(x_0+1)^{1.6}$

$T(n) \in O(n^{\log_2 3})$

$\Rightarrow T(n) \in O(n^{1.6})$

Karatsuba Guess and Check (Loose)

$$T(x_0+1) = 3T\left(\frac{x_0+1}{2}\right) + 8(x_0+1)$$ Recurrence definition

$$\leq 3\left(3000\left(\frac{x_0+1}{2}\right)^{1.6}\right) + 8(x_0+1)$$ Inductive hypothesis

$$\leq 3\left(3000\left(\frac{x_0+1}{2}\right)^{1.6}\right) + 8(x_0+1)^{1.6}$$ $\forall x \geq 0\colon x^{1.6} \geq x$

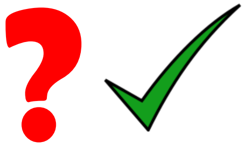
$$= \left(\frac{9000}{2^{1.6}} + 8\right)(x_0+1)^{1.6}$$ Distributive property

$$\leq 3000(x_0+1)^{1.6}$$ $\frac{9000}{2^{1.6}} + 8 \leq 3000$

Show: $T(x_0+1) \leq 3000(x_0+1)^{1.6}$

Recurrence Solving Techniques

Tree



Guess/Check

"Cookbook"

Substitution

Observation

Divide: $D(n)$ time

Conquer: Recurse on smaller problems of size $s_1, \ldots, s_k$

Combine: $C(n)$ time

Recurrence:

- $T(n) = D(n) + \sum_{i \in [k]} T(s_i) + C(n)$

Many divide and conquer algorithms have recurrences are of form:

- $T(n) = a \cdot T(n/b) + f(n)$

a and b are constants

Mergesort: $T(n) = 2T(n/2) + n$

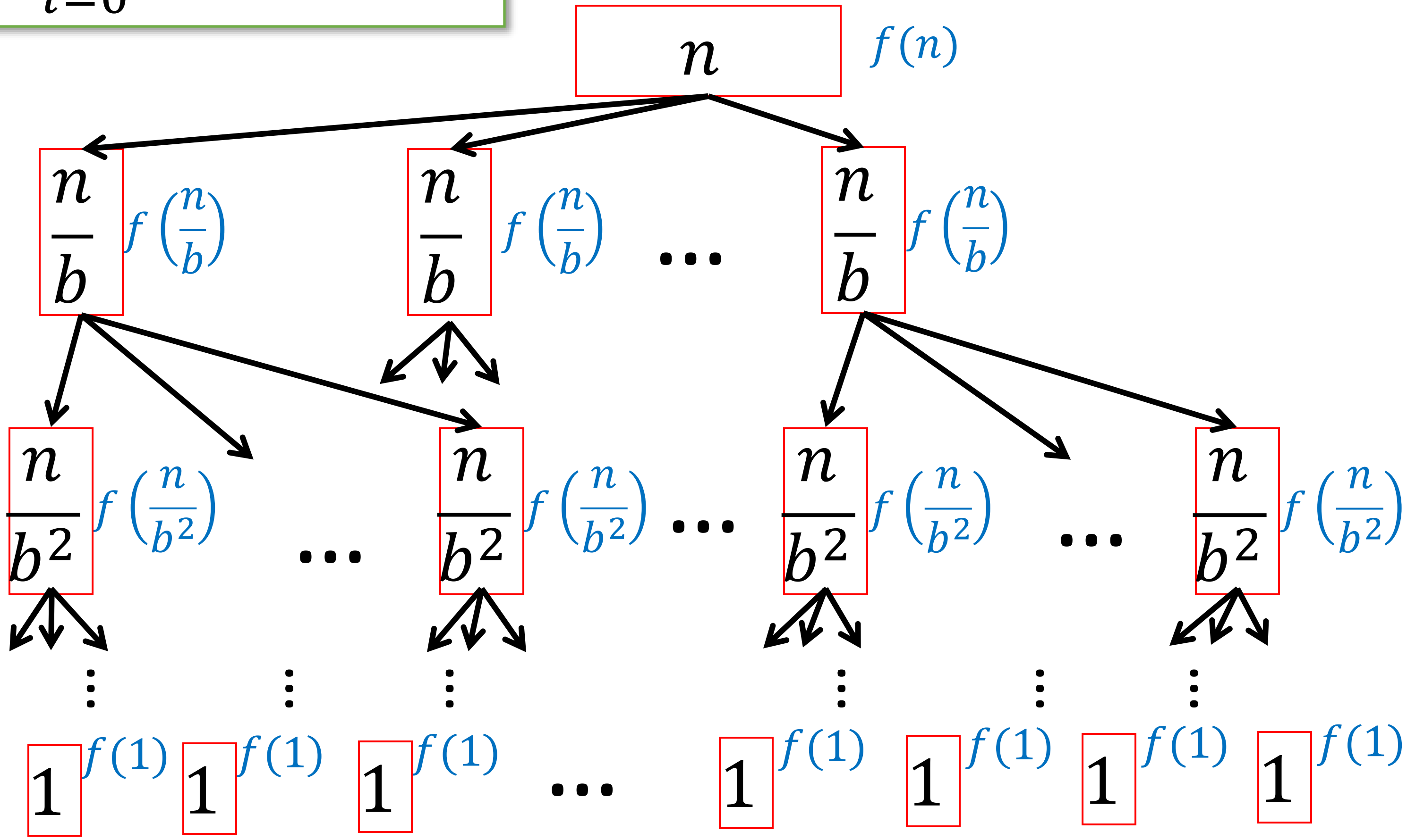
Divide and Conquer Multiplication: $T(n) = 4T(n/2) + 5n$

Karatsuba Multiplication: $T(n) = 3T(n/2) + 8n$

General Recurrence

$$T(n) = \sum_{i=0}^{\log_b n} a^i \cdot f\left(\frac{n}{b^i}\right)$$

$$T(n) = aT(n/b) + f(n)$$



k levels

$\log_b n$

Number of subproblems	Cost of subproblem
1	$f(n)$
a	$f(n/b)$
a^2	$f(n/b^2)$
a^k	$f(n/b^k)$

General Recurrence

$k = \log_b n$

$a = b^{\log_b a}$

An aside:

$a^{\log_b n} = \left(b^{\log_b a}\right)^{\log_b n} = \left(b^{\log_b n}\right)^{\log_b a} = n^{\log_b a}$

Three Cases

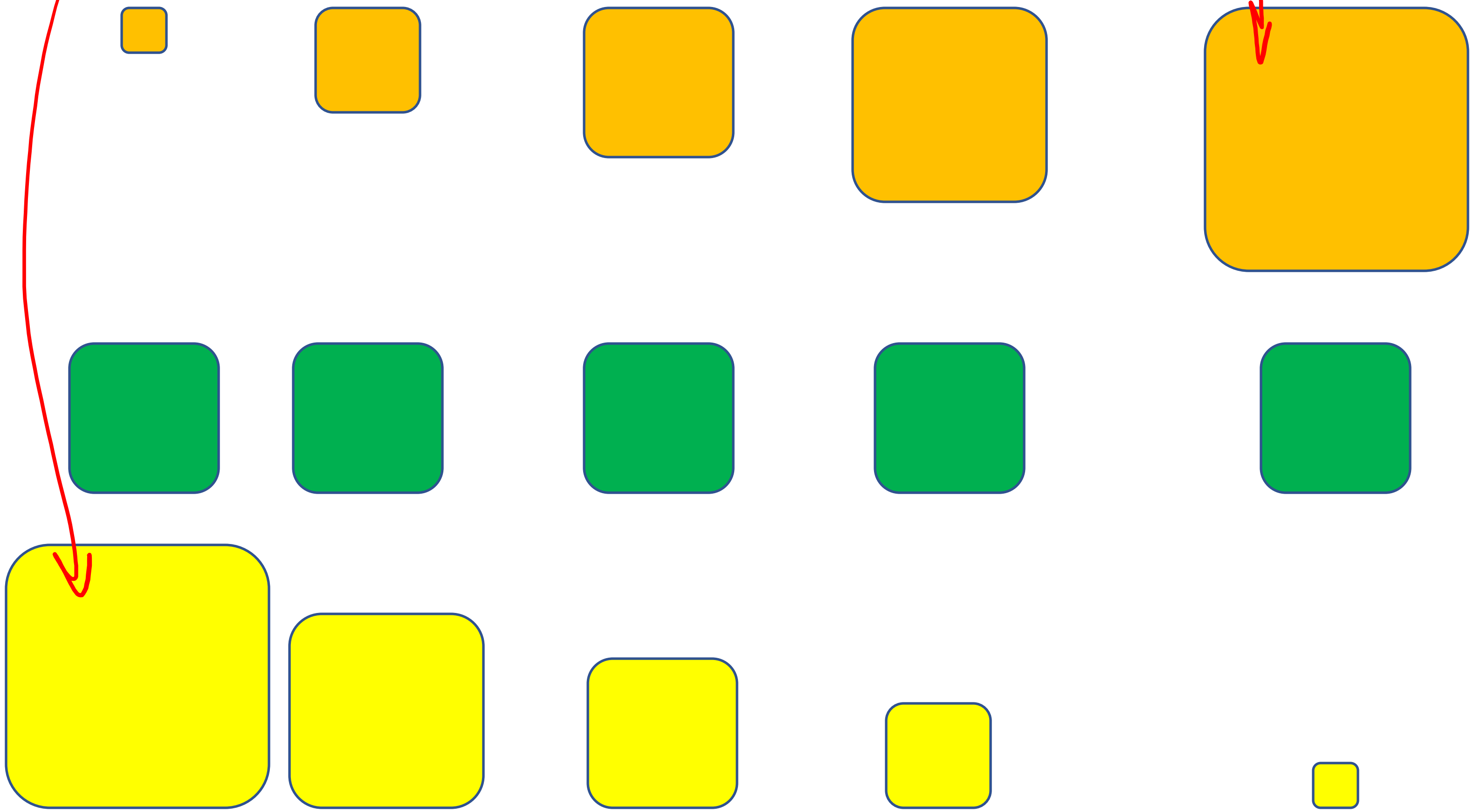
$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^k f\left(\frac{n}{b^k}\right)$$

$$k = \log_b n$$

Case 1:
Most work happens at the leaves

Case 2:
Work happens consistently throughout

Case 3:
Most work happens at top of tree



Master Theorem

$$T(n) = aT(n/b) + f(n)$$

$$\delta = \log_b a$$

	Requirement on f	Implication
Case 1	$f(n) \in O(n^{\delta-\varepsilon})$ for some constant $\varepsilon > 0$	$T(n) \in \Theta(n^{\delta})$

Master Theorem

$$T(n) = aT(n/b) + f(n)$$

$$\delta = \log_b a$$

	Requirement on f	Implication
Case 1	$f(n) \in O(n^{\delta-\varepsilon})$ for some constant $\varepsilon > 0$	$T(n) \in \Theta(n^{\delta})$
Case 2	$f(n) \in \Theta(n^{\delta})$	$T(n) \in \Theta(n^{\delta} \log n)$

Master Theorem

$$T(n) = aT(n/b) + f(n)$$

$$\delta = \log_b a$$

	Requirement on f	Implication
Case 1	$f(n) \in O(n^{\delta-\varepsilon})$ for some constant $\varepsilon > 0$	$T(n) \in \Theta(n^{\delta})$
Case 2	$f(n) \in \Theta(n^{\delta})$	$T(n) \in \Theta(n^{\delta} \log n)$
Case 3	$f(n) \in \Omega(n^{\delta+\varepsilon})$ for some constant $\varepsilon > 0$ **AND** $af\left(\frac{n}{b}\right) \leq cf(n)$ for constant $c < 1$ and sufficiently large n	$T(n) \in \Theta(f(n))$

Master Theorem Example 1

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

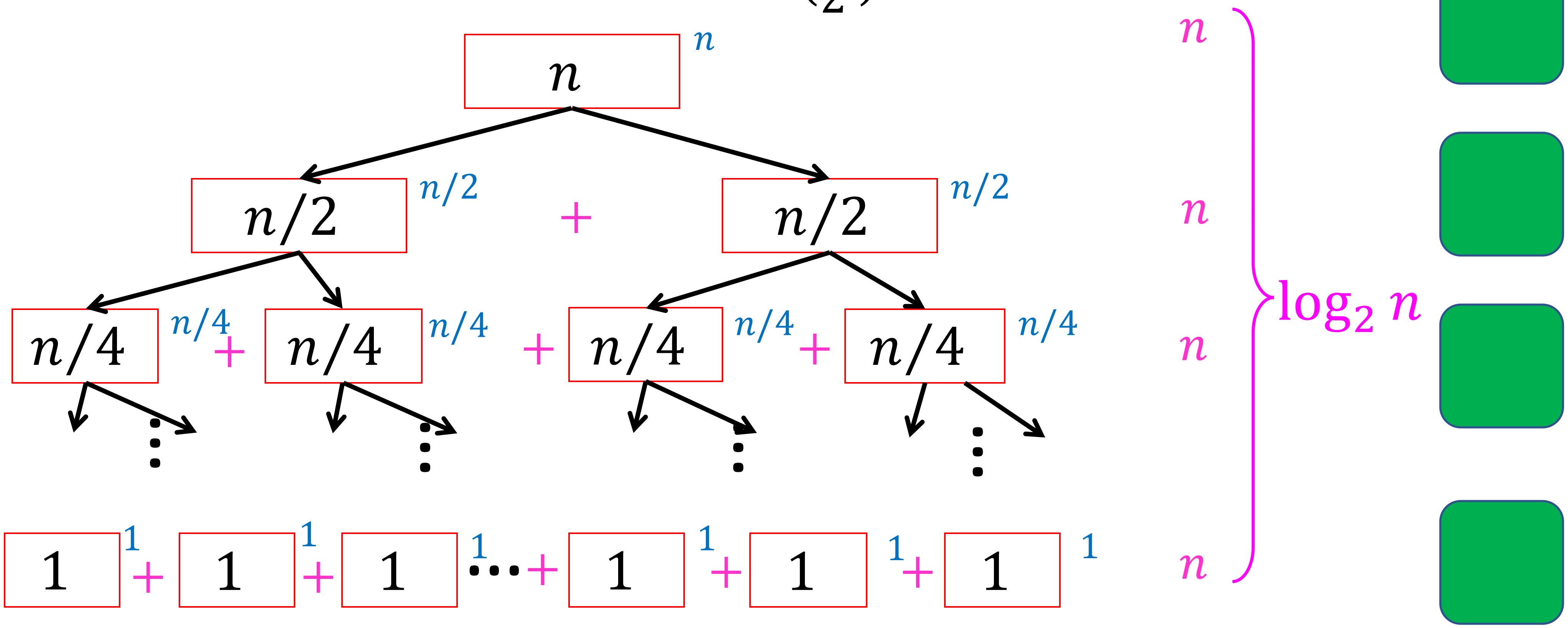
$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Case 2

$$\Theta\left(n^{\log_2 2} \log n\right) = \Theta(n \log n)$$

Tree method

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



Master Theorem Example 2

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

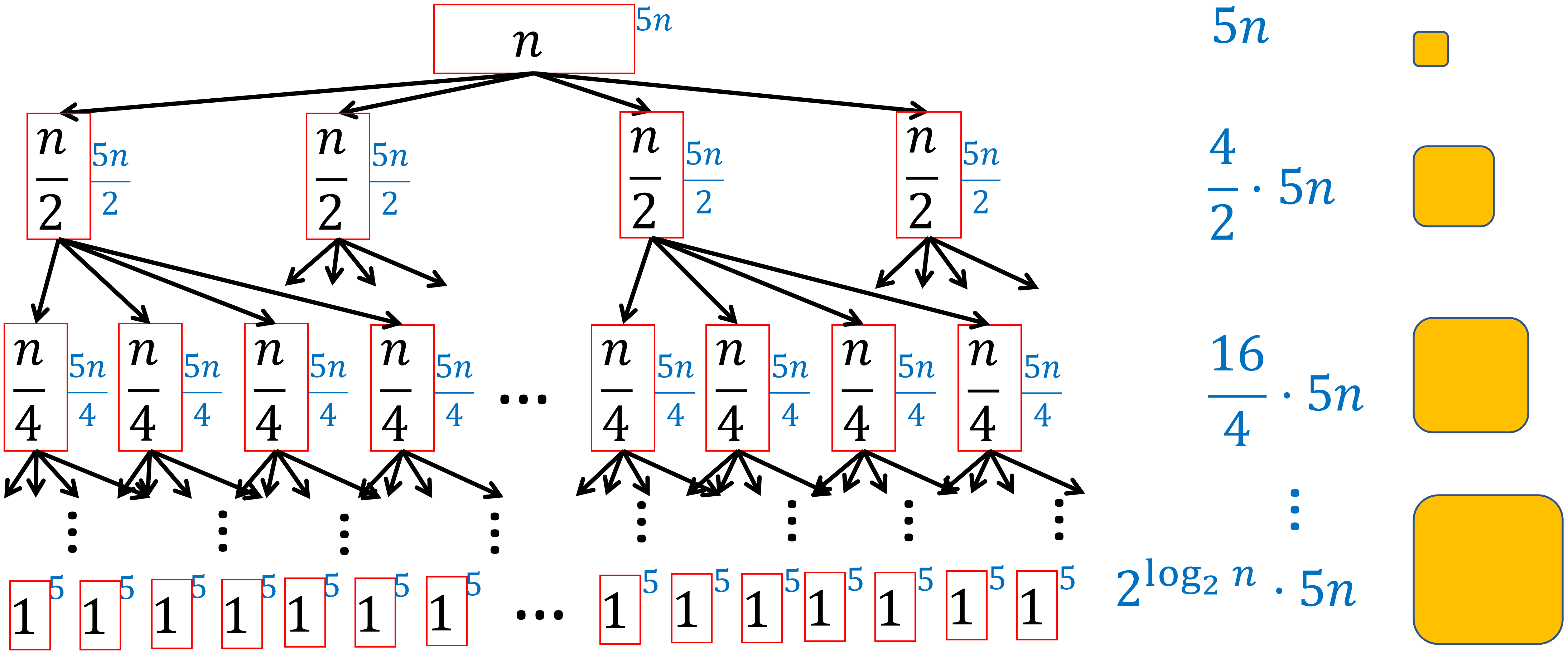
$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

Case 1

$$\Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

Tree method

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$



Tree method

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

Cost is <u>increasing</u> with the recursion depth
(due to large number of subproblems)

Most of the work happening in the leaves

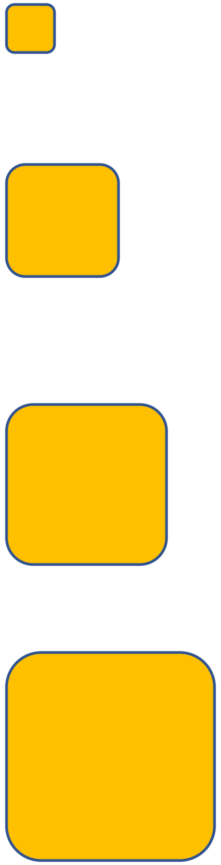
$5n$

$\frac{4}{2} \cdot 5n$

$\frac{16}{4} \cdot 5n$

$\vdots$

$2^{\log_2 n} \cdot 5n$



Master Theorem Example 3

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

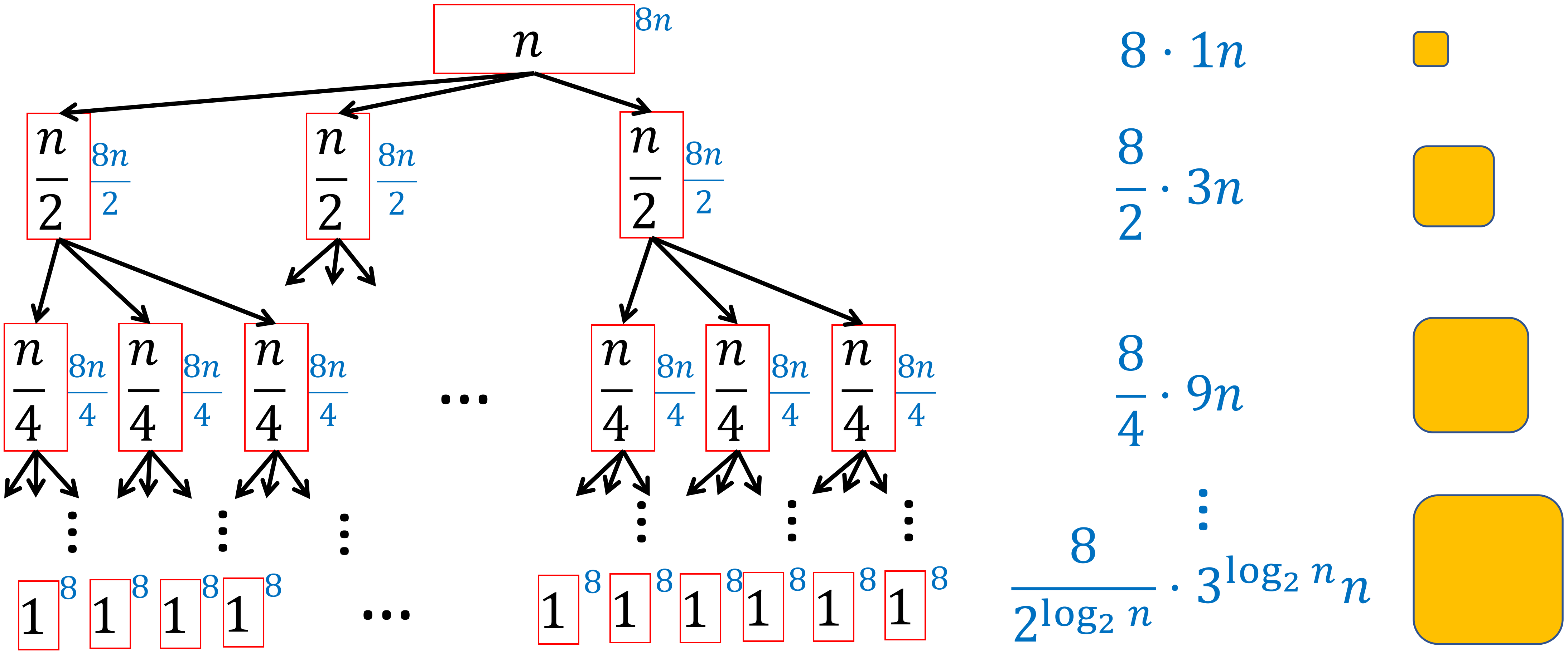
$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Case 1

$$\Theta\left(n^{\log_2 3}\right) \approx \Theta(n^{1.585})$$

Karatsuba

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$



Master Theorem Example 4

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$$T(n) = 2T\left(\frac{n}{2}\right) + 15n^3$$

Case 3

Master Theorem Example 4

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$$T(n) = 2T\left(\frac{n}{2}\right) + 15n^3$$

Case 3

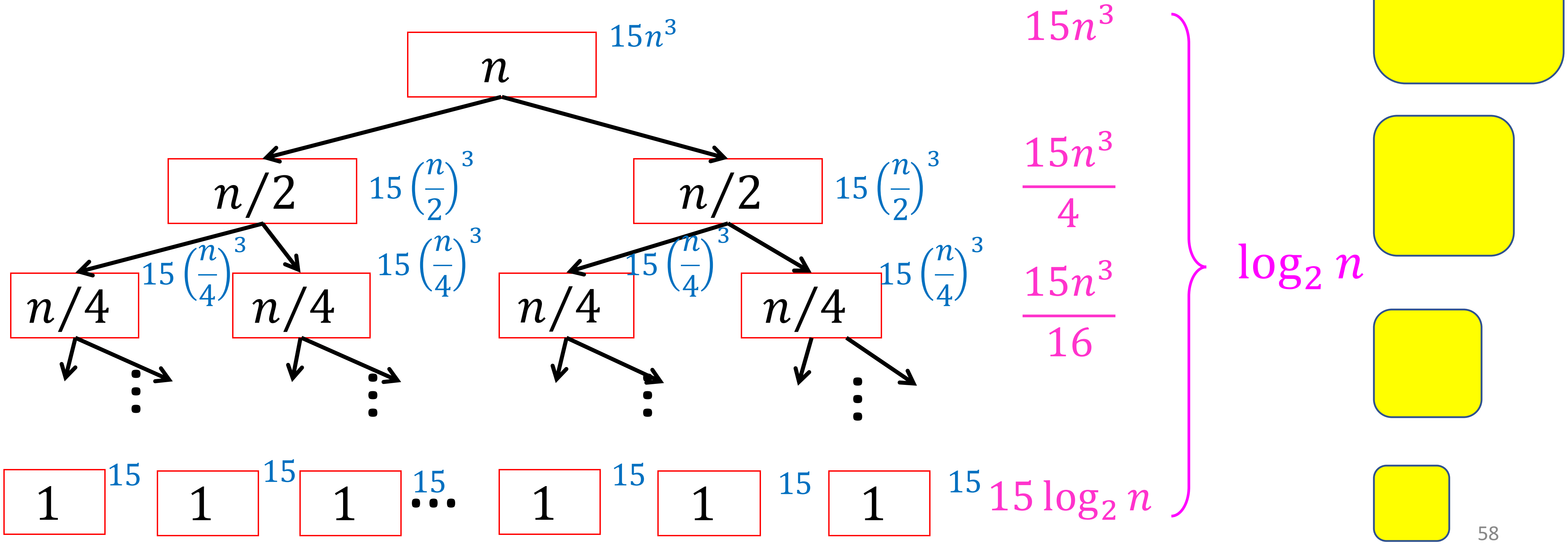
$\Theta(n^3)$

Important: For Case 3, need to additionally check that $2f(n/2) \le cf(n)$ for constant $c < 1$ and sufficiently large n

$$2f(n/2) = 30(n/2)^3 = \frac{30}{8}n^3 \le \frac{1}{4}(15n^3)$$

Master Theorem Example 4 (Visually)

$$T(n) = 2T(n/2) + 15n^3$$



Master Theorem Example 4 (Visually)

$$T(n) = 2T(n/2) + 15n^3$$

Cost is <u>decreasing</u> with the recursion depth (due to high *non-recursive* cost)

Most of the work happening at the top

$15n^3$

$\frac{15n^3}{4}$

$\frac{15n^3}{16}$

$15 \log_2 n$

$\log_2 n$