CS 3100
Data Structures and Algorithms 2
Lecture 13: Minimum Spanning Tree Algorithms

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Spring 2024

Readings in CLRS 4th edition:

- Chapter 21

Announcements

- PS5 due tomorrow
- PA3 coming soon!
- Grading update
  - PS0-2 grades returned, PS3 coming very soon
  - Regrade requests:
    - PS0-2 open through Sunday 3/17pm
    - PS3 and onward: 7 days after release
- Office hours (reminder)
  - Prof Hott Office Hours: Mondays 11a-12p, Fridays 10-11a and 2-3p
  - Prof Pettit Office Hours: Mondays and Fridays 2:30-4:00p
  - TA office hours posted on our website
  - Office hours are not for "checking solutions"

Reminders about Greedy Algorithms

Reminder: Some Terminology

Optimization problems: terminology

- A solution must meet certain constraints:
  A solution is *feasible*

Example: A possible shortest path must meet these criteria:
All edges must be in the graph and form a simple path.

- Solutions judged on some criteria:

Objective function

Example: The sum of edge weights in path is minimum

- One (or more) feasible solutions that scores highest (by the objective function) is called the *optimal solution(s)*

The **greedy approach** is often a good choice for optimization problems

- So is **dynamic programming** (coming later in the course)

Reminder: Greedy Strategy: An Overview

Greedy strategy:

- Build solution by stages, adding one item to the partial solution we've found before this stage
- At each stage, make *locally optimal choice* based on the ***greedy choice*** (sometimes called the *greedy rule* or the *selection function)*
  - Locally optimal, i.e. best given what info we have now
- Irrevocable: a choice can't be un-done
- Sequence of locally optimal choices leads to globally optimal solution (hopefully)
  - Must prove this for a given problem!

Reminder: We've Seen Greedy Graph Algorithms

Dijkstra's Shortest Path is greedy!

Build solution by adding item to partial solution

- Dijkstra's: add edge to connect kth vertex, where the edges for the $k-1$ already selected show the shortest paths to those $k-1$ vertices

Greedy choice

- Dijkstra's: for all vertices connected to one of the $k-1$ vertices already processed, choose w where $dist(s,w)$ is the minimum

We did have to prove that this sequence of locally optimal choices leads to globally optimal solution

Summary of the Greedy Approach

Problem must have Optimal Substructure

- Optimal solution to a problem contains optimal solutions to subproblems

Idea:

1. Identify a greedy choice property
   - How to make a choice guaranteed to be included in some optimal solution
2. Repeatedly apply the choice property until no subproblems remain

Greedy approach only considers one subproblem at each stage

Change Making Choice Property

Our algorithm's **Greedy choice:**
Choose largest coin less than or equal to target value

Leads to optimal solution?

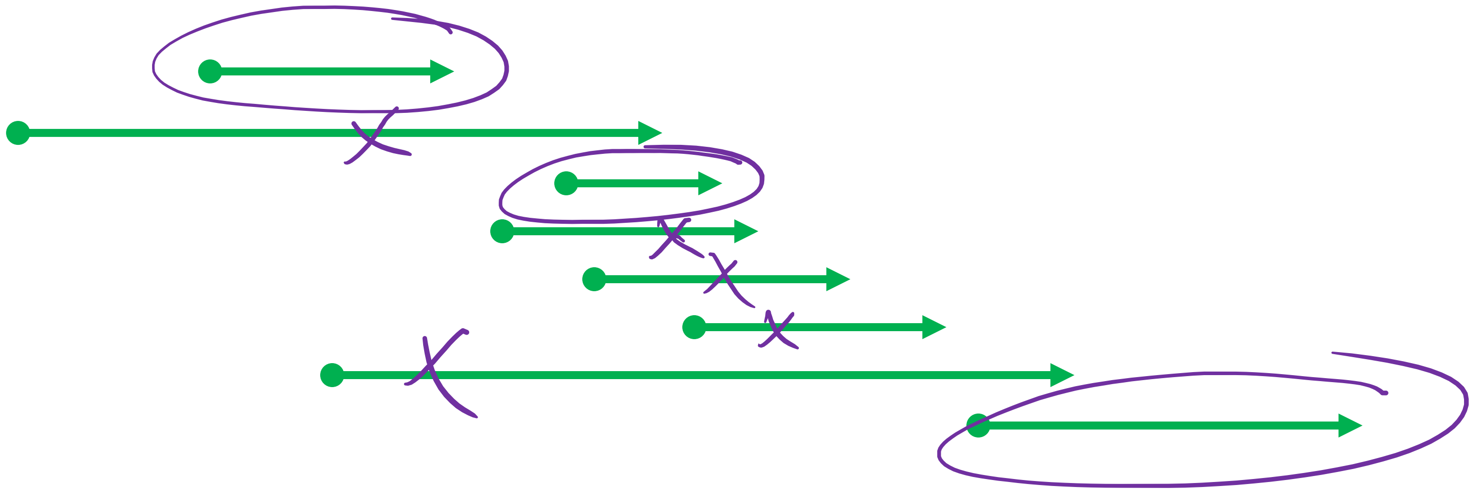
- For standard U.S. coins: Yes, coin chosen must be part of some optimal solution. We can prove it!
- For "unusual" sets of coins? We saw a counter-example.
- For U.S. postage stamps? Hmm…

Interval Scheduling Algorithm

Find event ending earliest, add to solution,

Remove it and all conflicting events,

Repeat until all events removed, return solution



Interval Scheduling Run Time

Find event ending earliest, add to solution,

Remove it and all conflicting events,

Repeat until all events removed, return solution

```
Sort intervals by finish time

StartTime = 0
for each interval (in order of finish time):
        if begin of interval > StartTime:
                add interval to solution
                StartTime = end of interval
```

$\Theta(n \log n)$

$\Theta(n)$

$\Theta(n \log n)$

Exchange argument

Shows correctness of a greedy algorithm

Idea:

- Show exchanging an item from an arbitrary optimal solution with your greedy choice makes the new solution no worse
- How to show my sandwich is at least as good as yours:
  - Show: “I can remove any item from your sandwich, and it would be no worse by replacing it with the same item from my sandwich”

Exchange Argument for Earliest End Time

Claim: earliest ending interval is always part of <u>some</u> optimal solution

0 — n

Let $OPT_{i,j}$ be an optimal solution for time range $[i, j]$

Let a^* be the first interval in $[i, j]$ to finish overall ← Greedy choice

If $a^* \in OPT_{i,j}$ then claim holds

Else if $a^* \notin OPT_{i,j}$, let a be the first interval to end in $OPT_{i,j}$

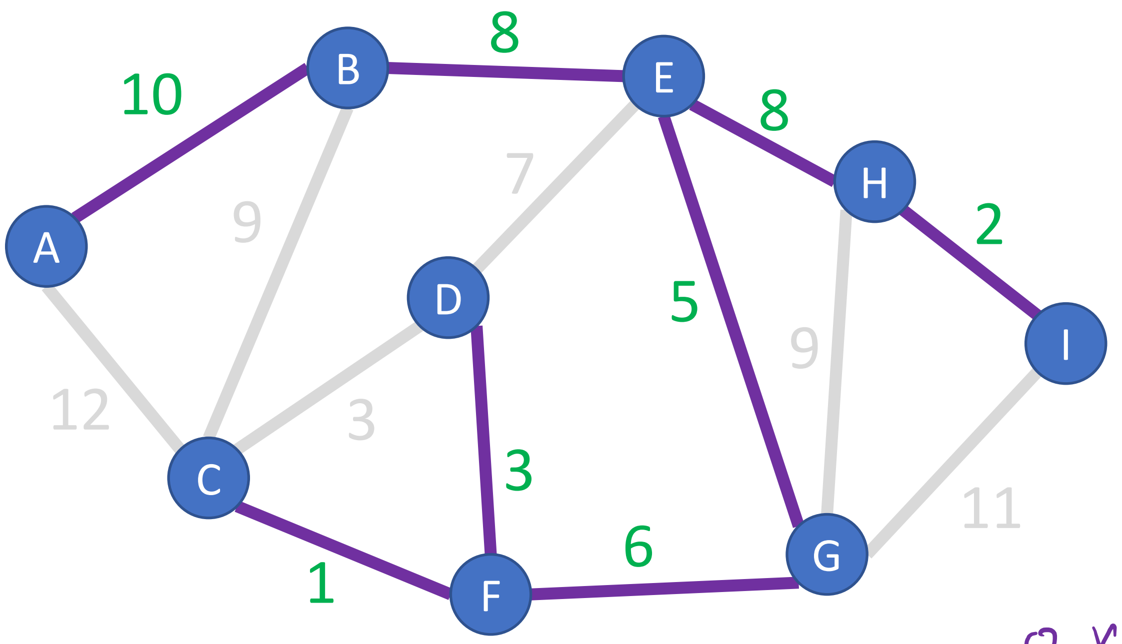
- By definition a^* ends before a, and therefore does not conflict with any other events in $OPT_{i,j}$

- Therefore $OPT_{i,j} - \{a\} + \{a^*\}$ is also an optimal solution
- Thus claim holds

Minimum Spanning Trees

Readings: CLRS 21
(but not 21.1)

Spanning Tree



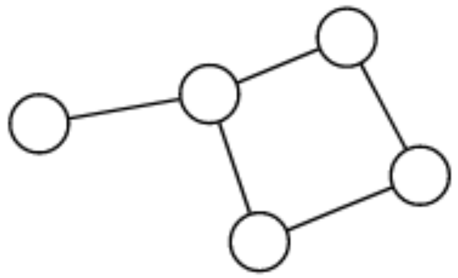
- All connected graphs have spanning tree(s)
- All spanning trees have the same number of nodes (all of them)
- You can construct a spanning tree by arbitrarily remove edges from cycles

How many edges does T have?

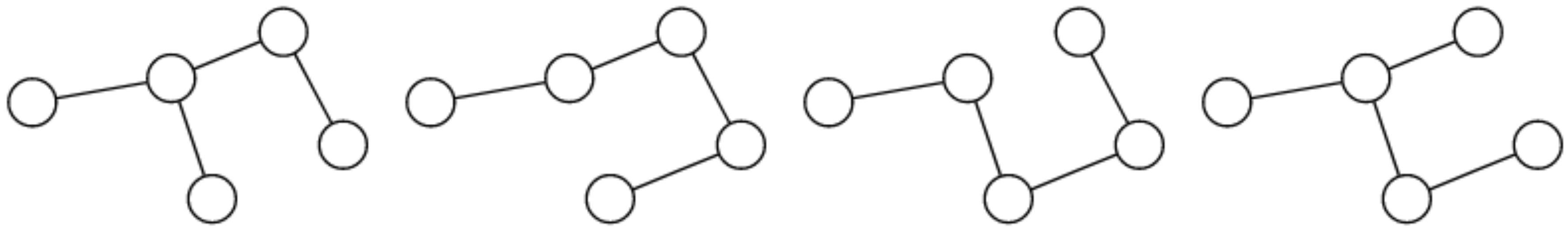
A tree $T = (V_T, E_T)$ is a **spanning tree** for an undirected graph $G = (V, E)$ if $V_T = V$, $E_T \subseteq E$ (namely, T connects or "spans" all the nodes in G)

Spanning Tree: Example

Original Graph:



Possible spanning trees:



Minimum Spanning Tree

Just constructing any spanning tree is simple

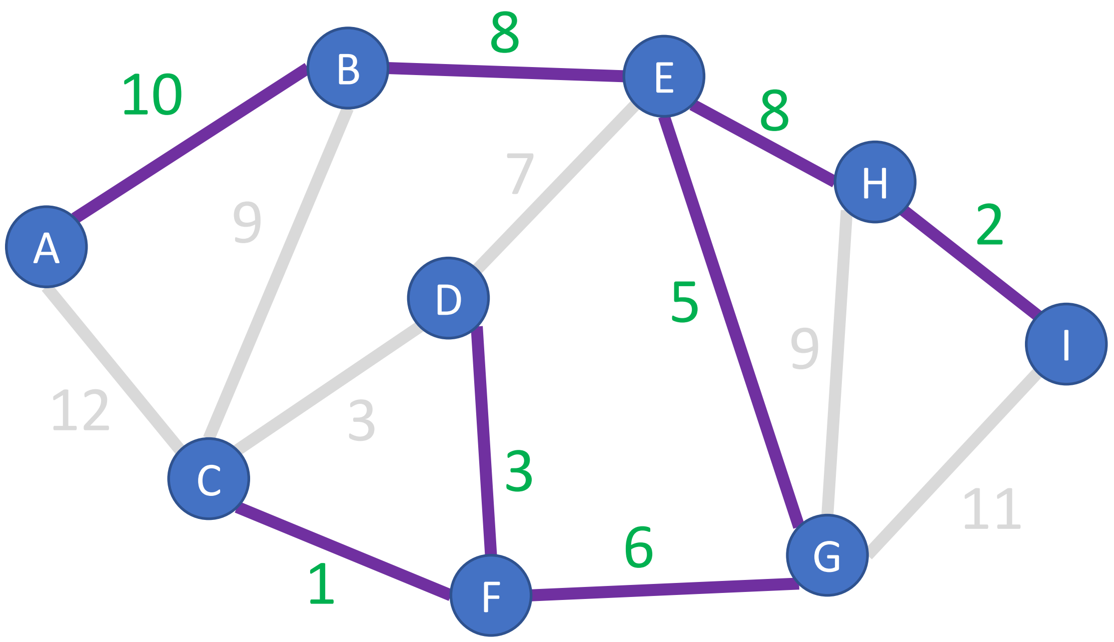
Suppose edges have weights

- Cost of building tracks between two stations
- Distance between two intersections/stops
- Length of wire between boxes in a house
- Cost to connect two nodes in a network

Each spanning tree has a different total **cost** (sum of edge weights included in tree)

The ***Minimum Spanning Tree*** is the spanning tree with lowest overall cost

Minimum Spanning Tree



$$\text{Cost}(T) = \sum_{e \in E_T} w(e)$$

How many edges does T have?

A tree $T = (V_T, E_T)$ is a **minimum spanning tree** for an undirected graph $G = (V, E)$ if T is a spanning tree of minimal cost

MST Algorithms

We'll see two greedy algorithms to find a graph's MST

- Prim's algorithm
  - Very similar to Dijkstra's SP algorithm
  - Builds a single tree, adding one edge to grow the tree
- Kruskal's algorithm
  - In a *forest* of trees, add an edge at each step to grow one tree or to connect two trees (don't make a cycle)
  - Utilizes an interesting data structure for manipulating sets

Prim’s Algorithm

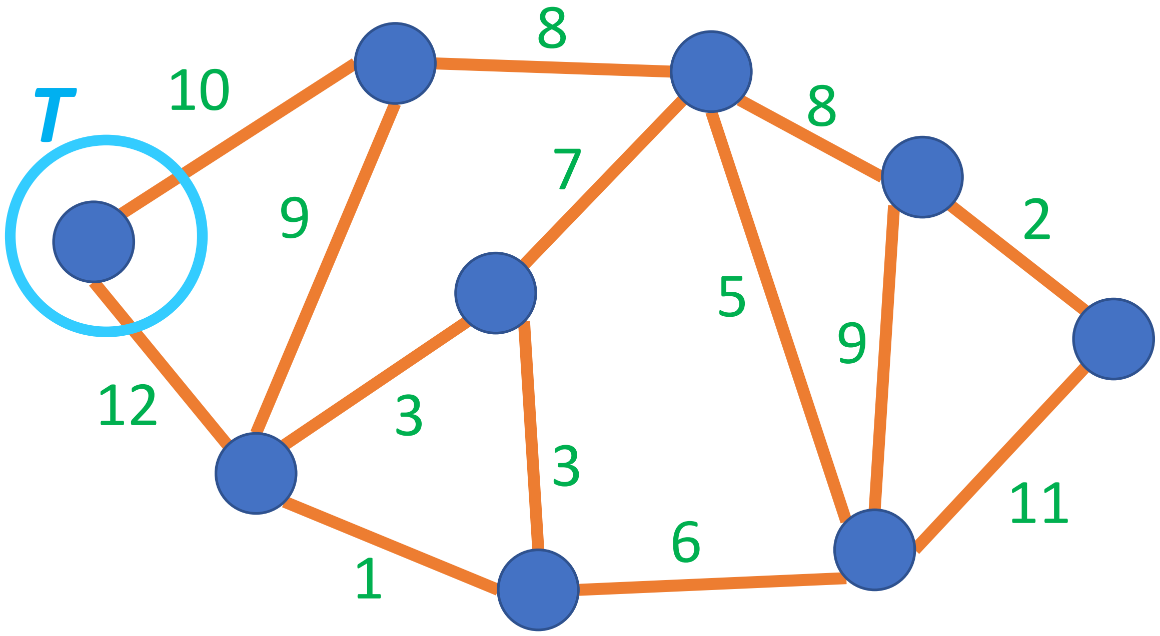
CLRS in 21.2

Reminder: Dijkstra's SP Algorithm

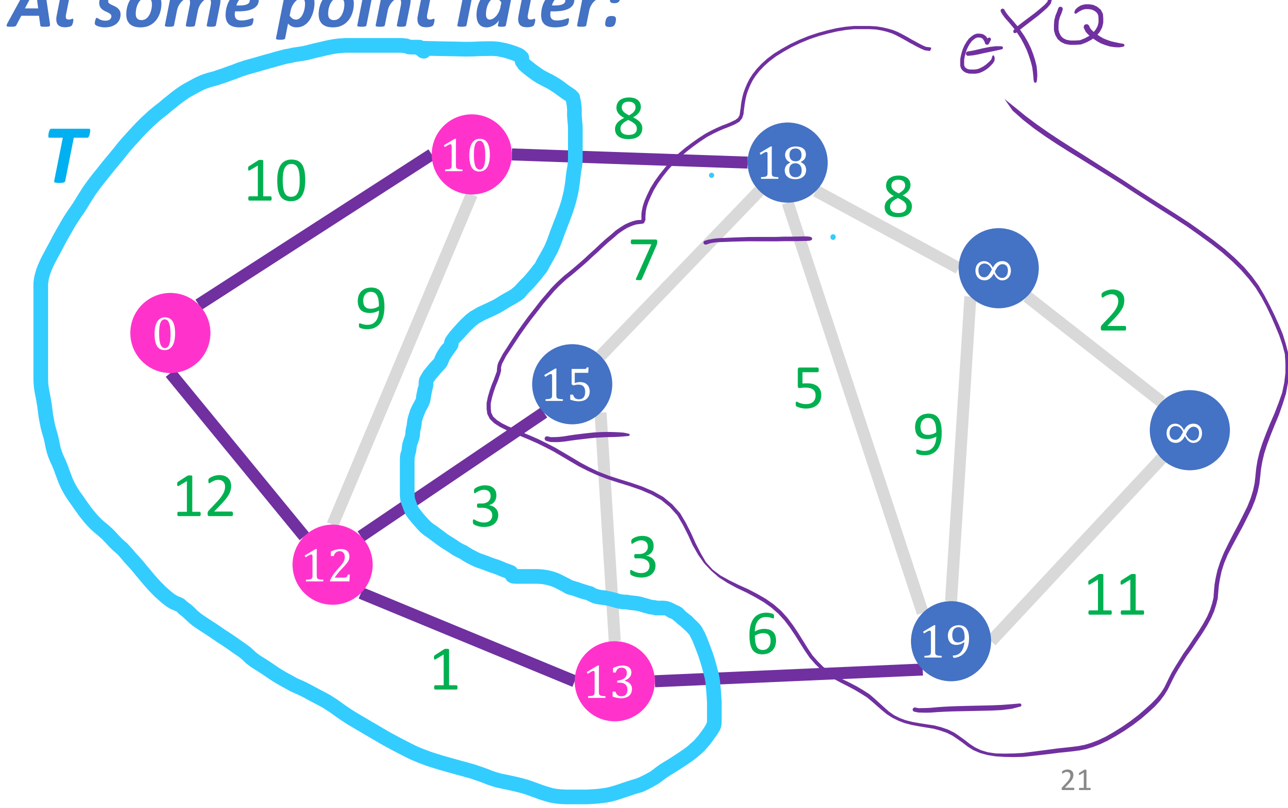
1. Start with an empty tree T and add the source to T
2. Repeat $|V| - 1$ times:
   - At each step, add the node **"nearest" to the source into tree T**

Greedy Choice Property!

Initially:



At some point later:

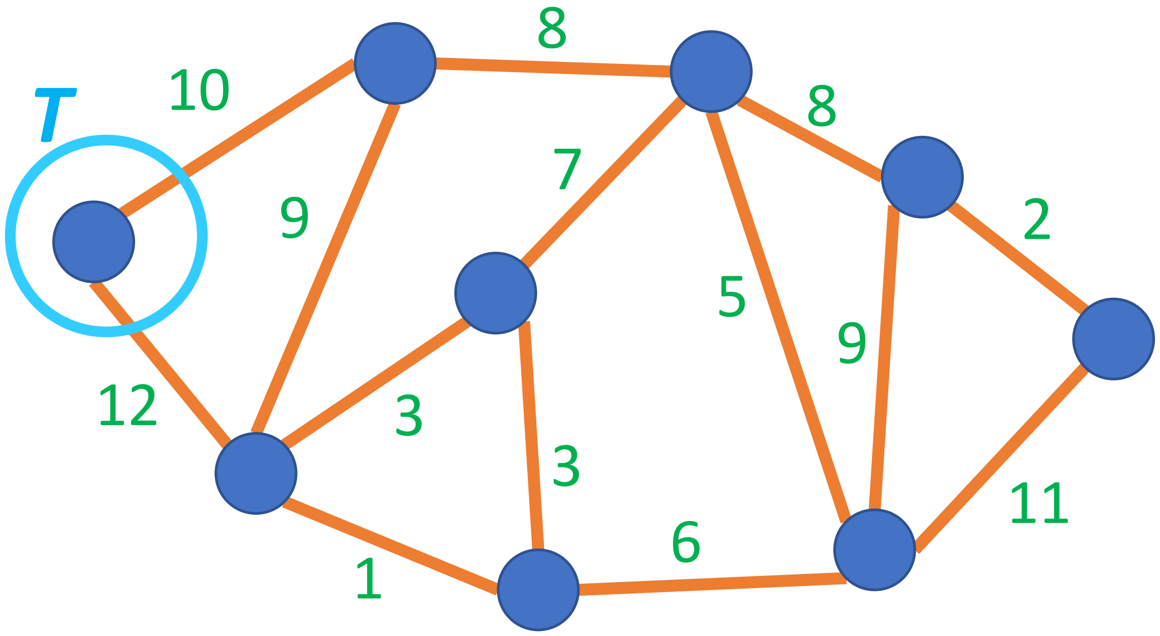


Prim's MST Algorithm

The *Greedy Choice*! Same strategy, but different greedy choice to solve a different problem

1. Start with an empty tree T and add the source to T
2. Repeat $|V| - 1$ times:
   - At each step, add the node with **minimum connecting edge to a node in T**

Initially:



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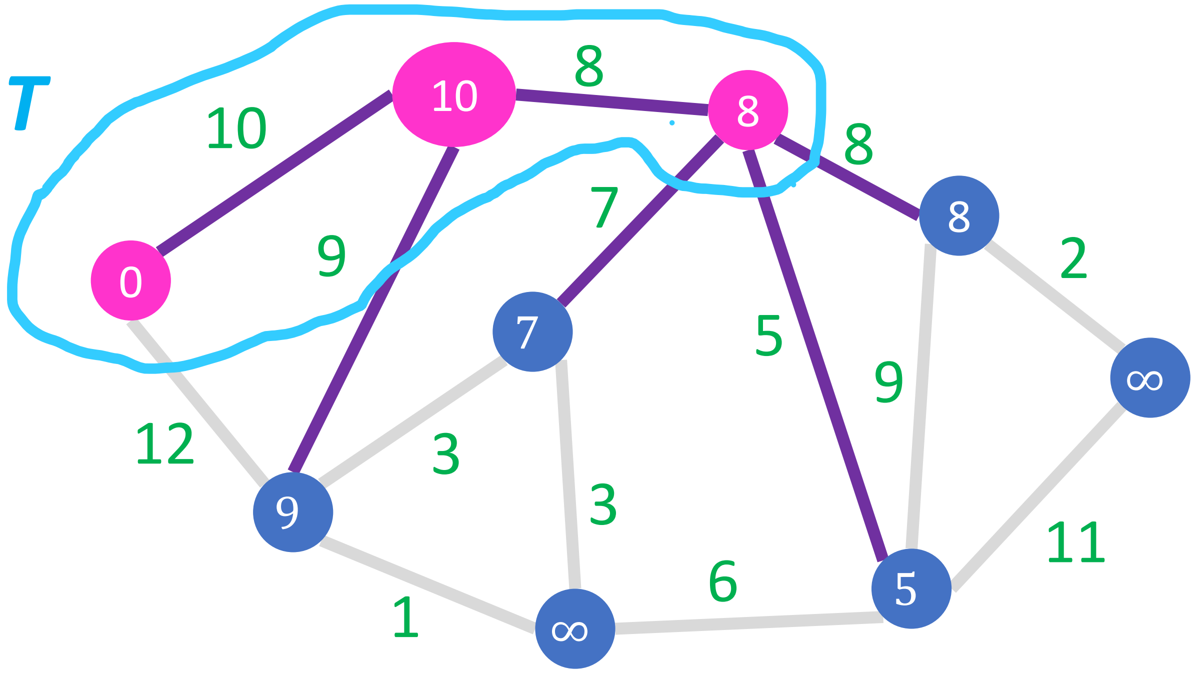
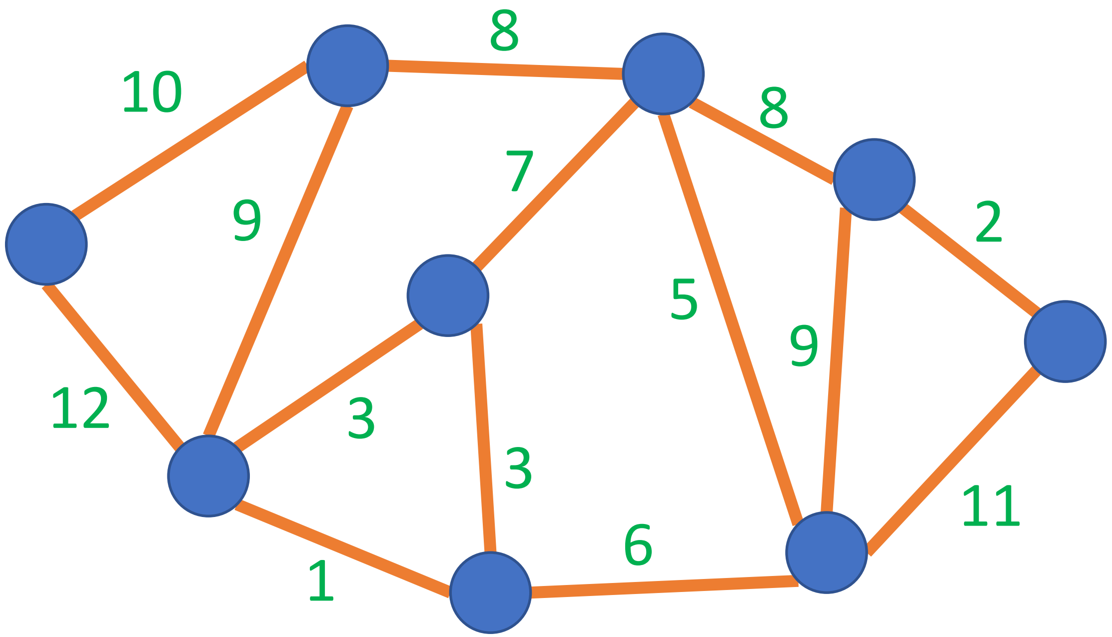


Prim's Algorithm

1. Start with an empty tree T and pick a start node and add it to T
2. Repeat $|V| - 1$ times:
   - Add the min-weight edge which connects a node in T with a node not in T



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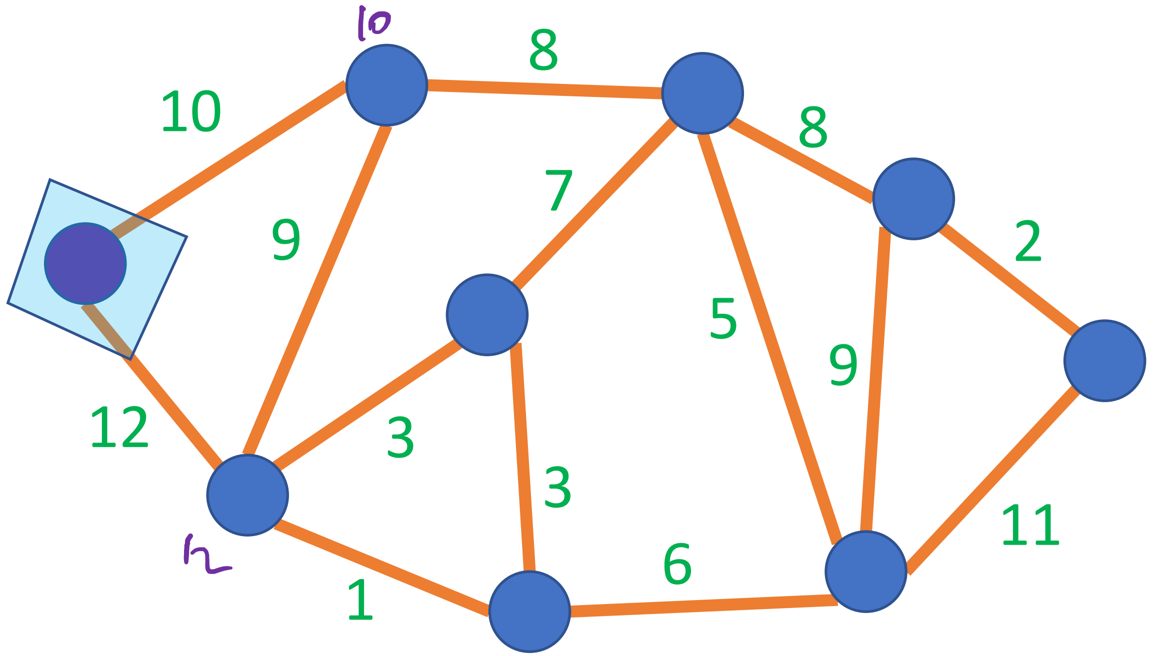

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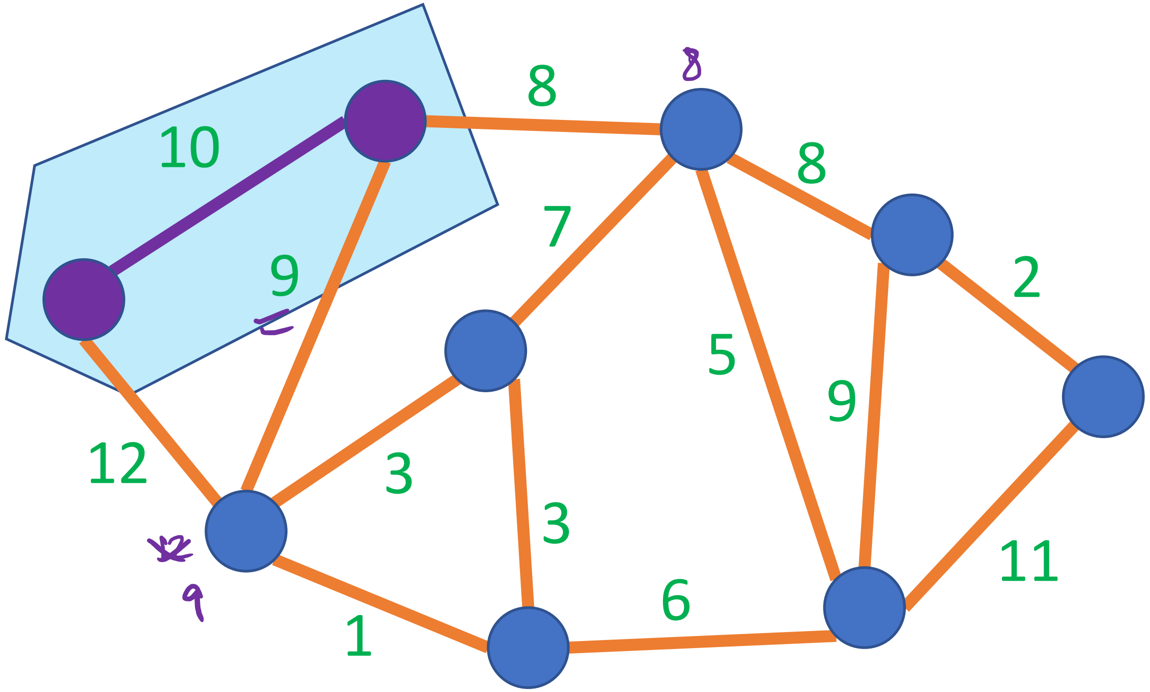

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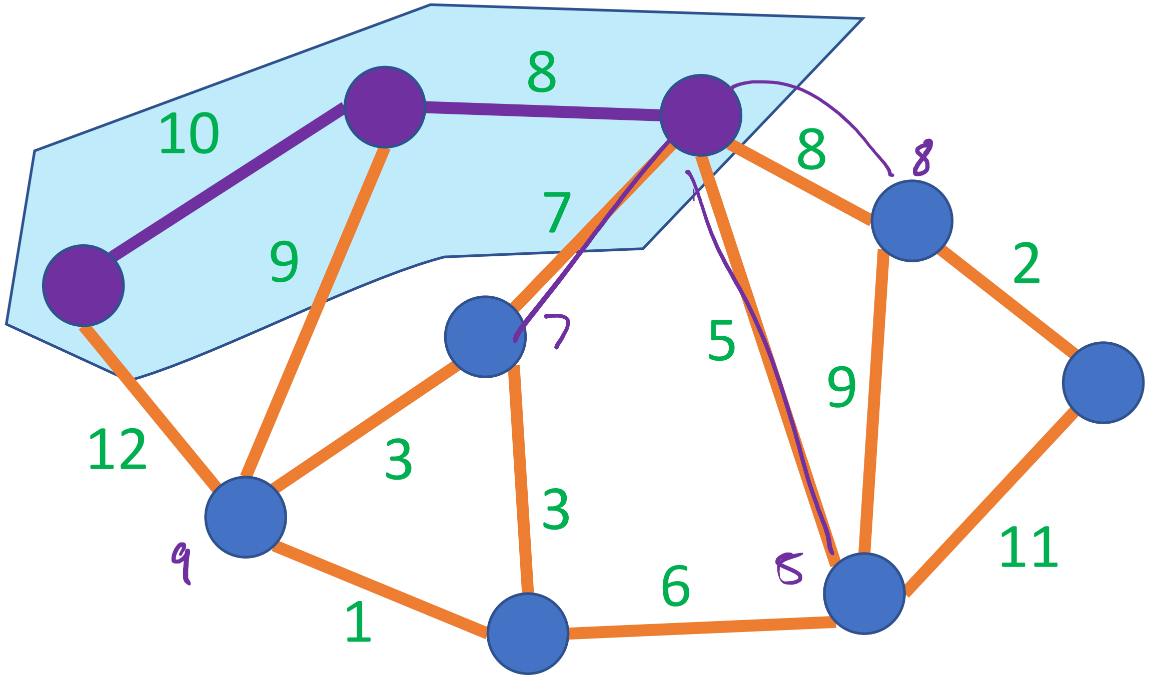

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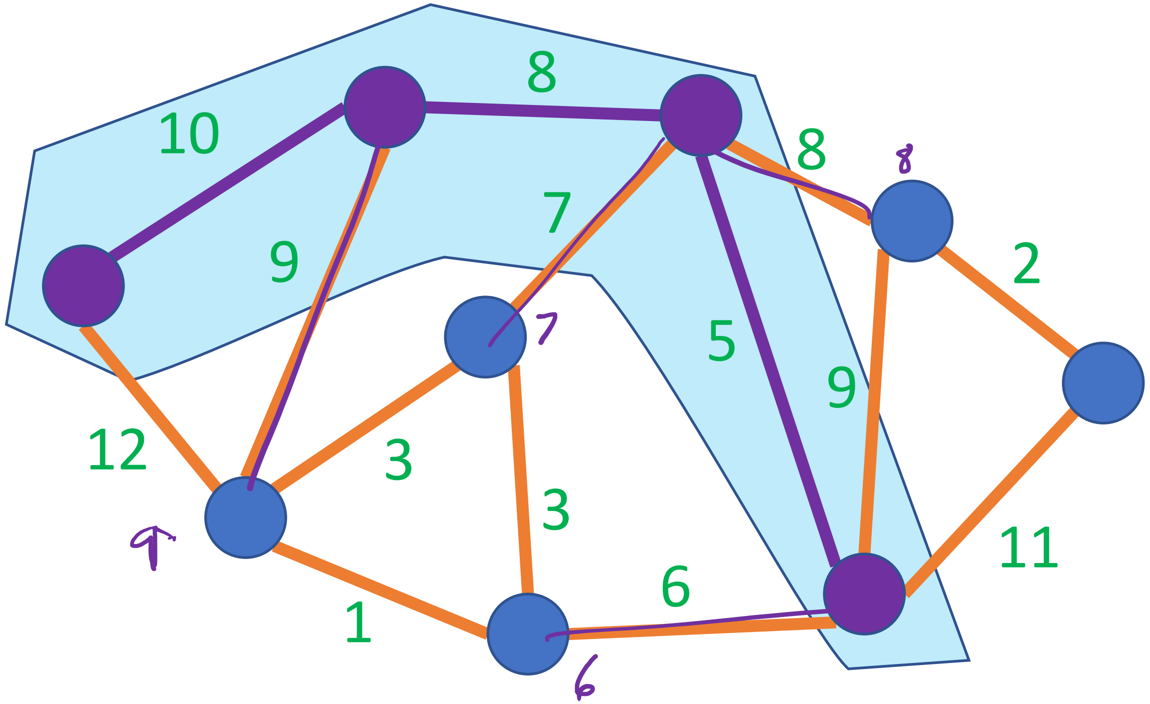

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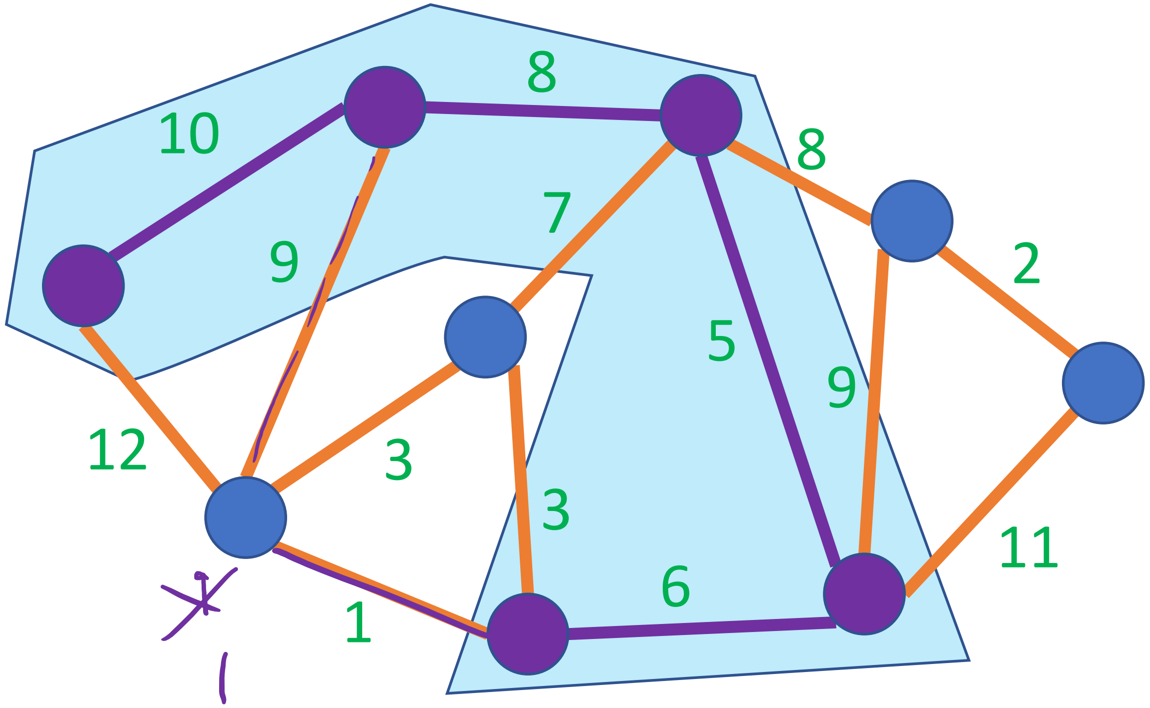

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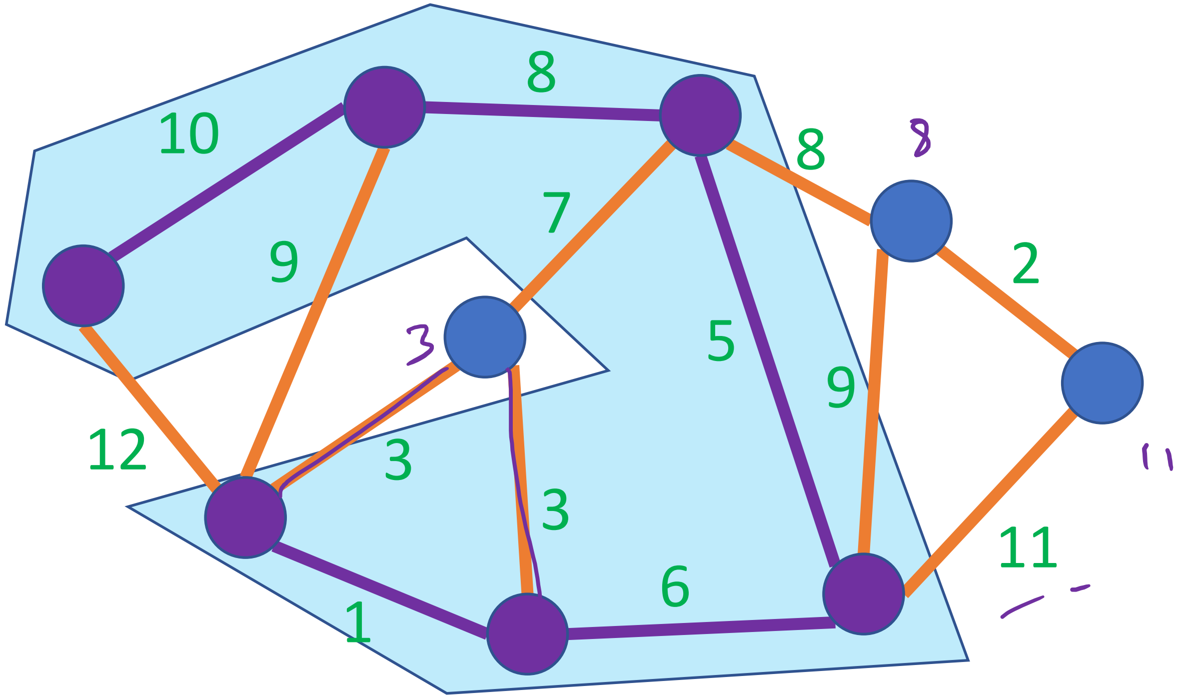

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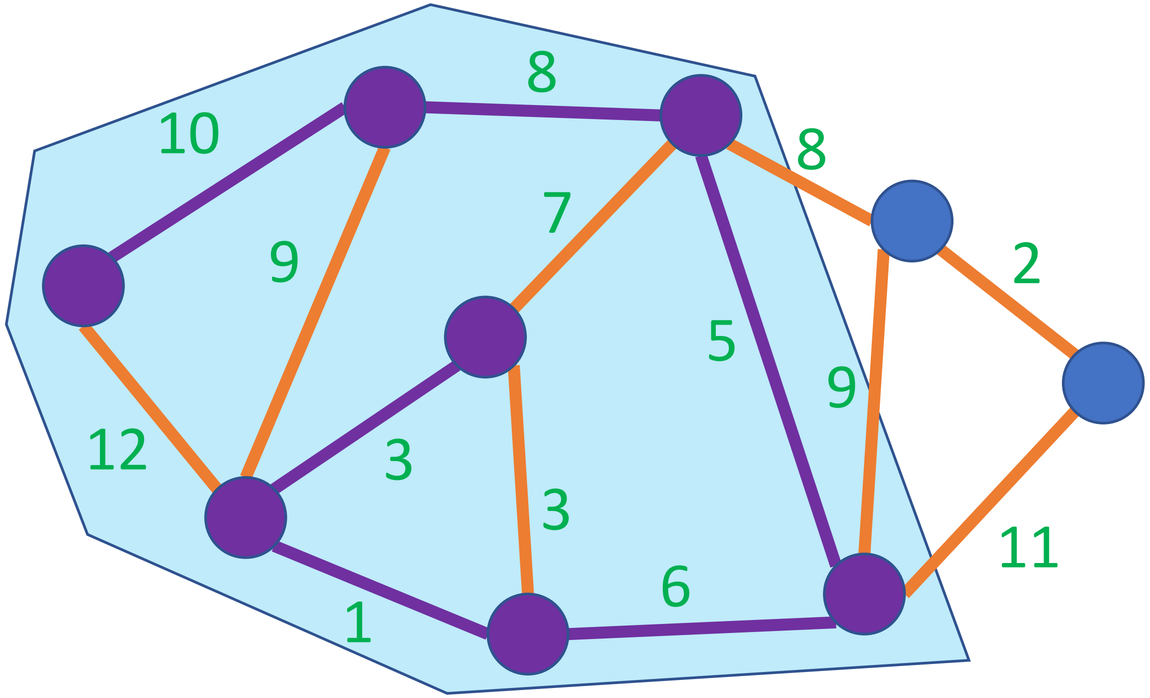

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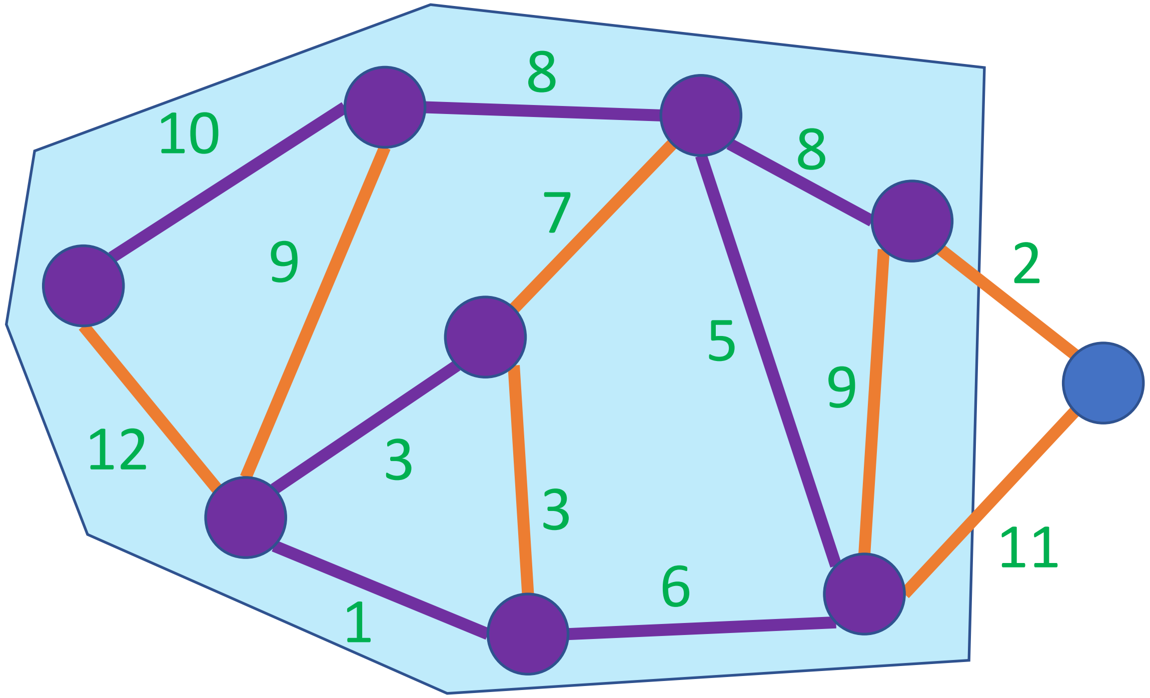

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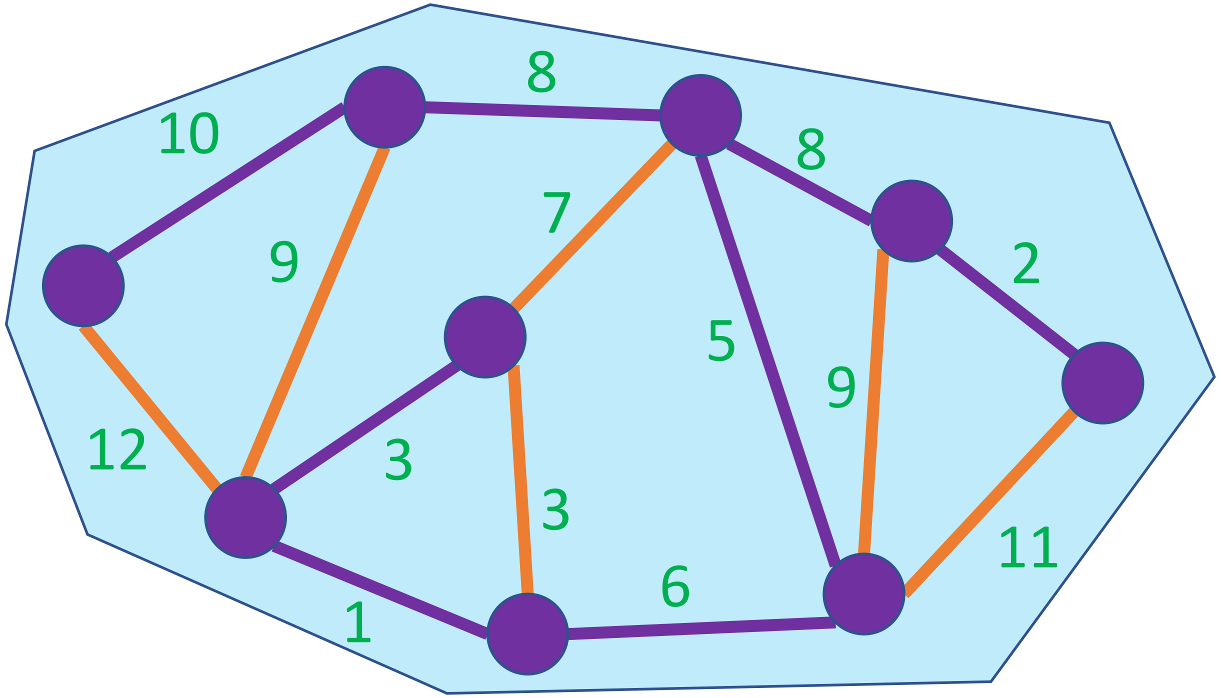

Prim's Algorithm

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Implementation:

- Maintain nodes **not in** T in a min-heap (priority queue)
- Find the next closest node v by extracting min from priority queue
- Each time node v is added to the tree, update keys for neighbors still in min-heap
- Repeat until no nodes left in min-heap

Prim's Algorithm Implementation

1. Start with an empty tree T and pick a start node and add it to T
2. Repeat $|V| - 1$ times:
   - Add the min-weight edge which connects a node in T with a node not in T

Implementation:

initialize $d_v = \infty$ for each node v
add all nodes $v \in V$ to the priority queue PQ, using d_v as the key
pick a starting node s and set $d_s = 0$
while PQ is not empty:
    $v = \text{PQ}.\text{extractMin}()$
    for each $u \in V$ such that $(v, u) \in E$:
        if $u \in \text{PQ}$ and $w(v, u) < d_u$:
            $\text{PQ}.\text{decreaseKey}(u, w(v, u))$
            $u.\text{parent} = v$

each node also maintains a parent, initially NULL

key: minimum cost to connect u to nodes in PQ

Prim's Algorithm

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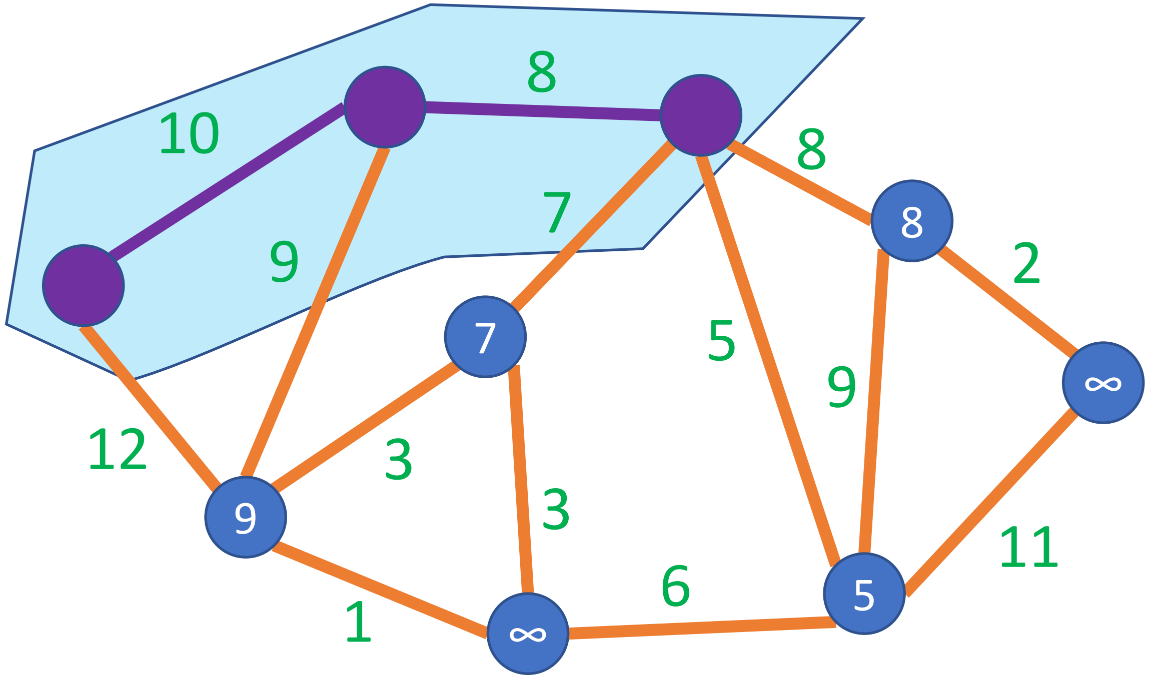

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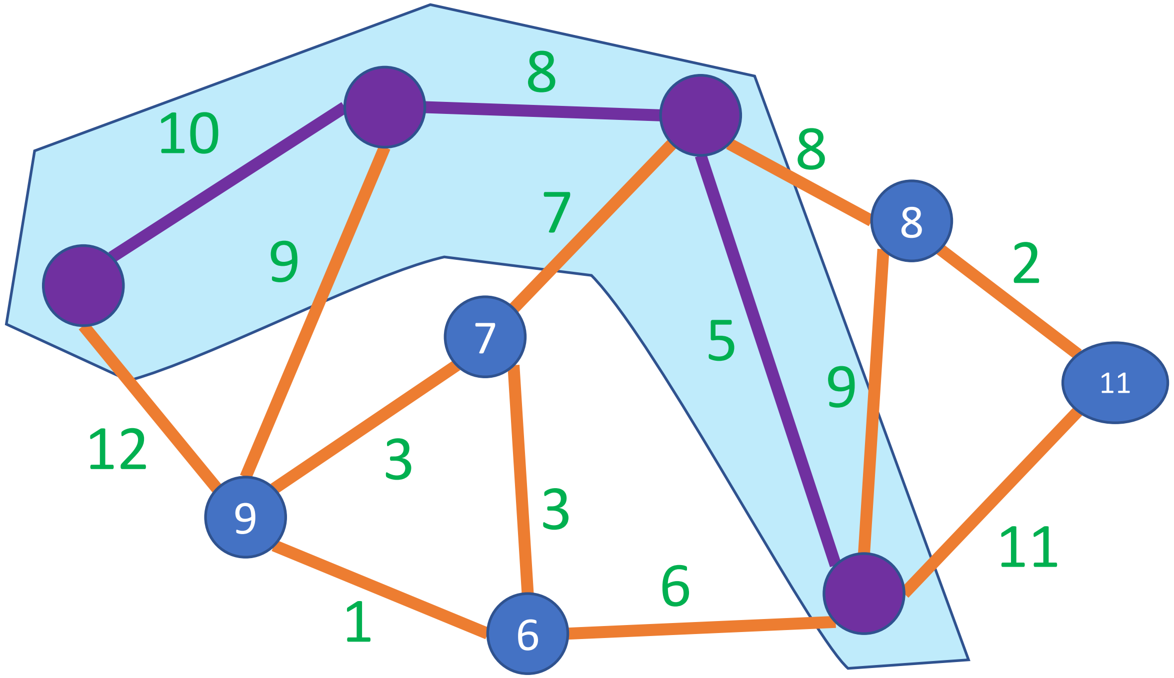

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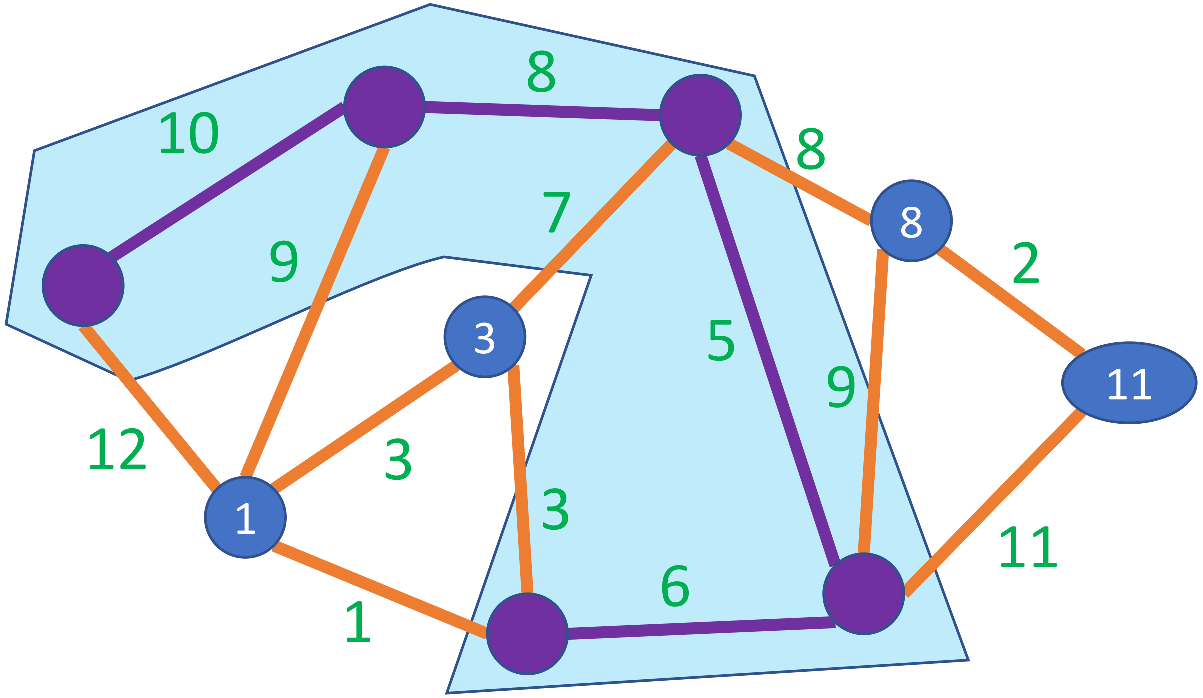

Reminder: Dijkstra's Algorithm Implementation

1. Start with an empty tree T and add the source to T
2. Repeat $|V| - 1$ times:
   - Add the "nearest" node not yet in T to T

Implementation:

initialize $d_v = \infty$ for each node v
add all nodes $v \in V$ to the priority queue PQ, using d_v as the key
set $d_s = 0$
while PQ is not empty:
  $v = \mathrm{PQ}.\mathrm{extractMin}()$
  for each $u \in V$ such that $(v, u) \in E$:
    if $u \in \mathrm{PQ}$ and $d_v + w(v, u) < d_u$:
      $\mathrm{PQ}.\mathrm{decreaseKey}\big(u, d_v + w(v, u)\big)$
      $u.\mathrm{parent} = v$

each node also maintains a parent, initially `NULL`

key: length of shortest path $s \to u$ using nodes in PQ

Prim's Algorithm Implementation

1. Start with an empty tree T and pick a start node and add it to T
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each node also maintains a parent, initially `NULL`

key: minimum cost to connect u to nodes in PQ

Prim's Algorithm Running Time

Same as for Dijkstra's Shortest Path algorithm!

Implementation (with nodes in the priority queue):

initialize $d_v = \infty$ for each node v
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pick a starting node s and set $d_s = 0$
while PQ is not empty:
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Initialization:
$O(|V|)$

$|V|$ iterations
$O(\log|V|)$

$|E|$ iterations total

$O(\log|V|)$

Using indirect heaps

Overall running time: $O(|V|\log|V| + |E|\log|V|) = O(|E|\log|V|)$

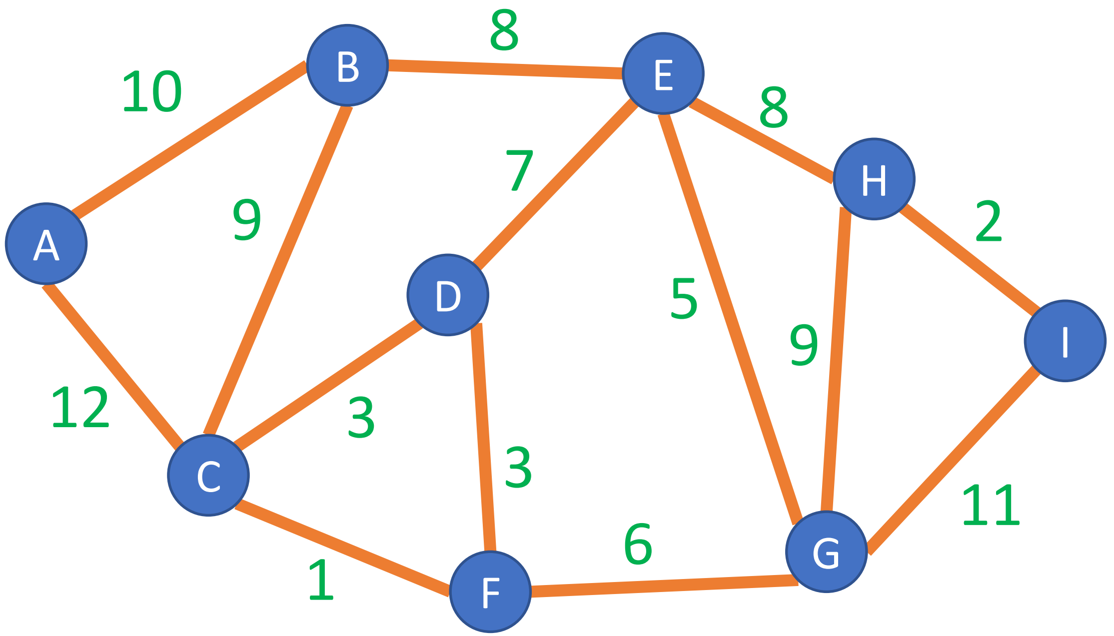
Kruskal’s MST Algorithm

Readings: CLRS first part of 21.2

Kruskal's Algorithm

The *Greedy Choice* for Kruskal's

1. Start with an empty set of edges T
2. Repeatedly add to T the lowest-weight edge that does not create a cycle. (Stop when we've added $\cancel{n}\, V - 1$ edges.)



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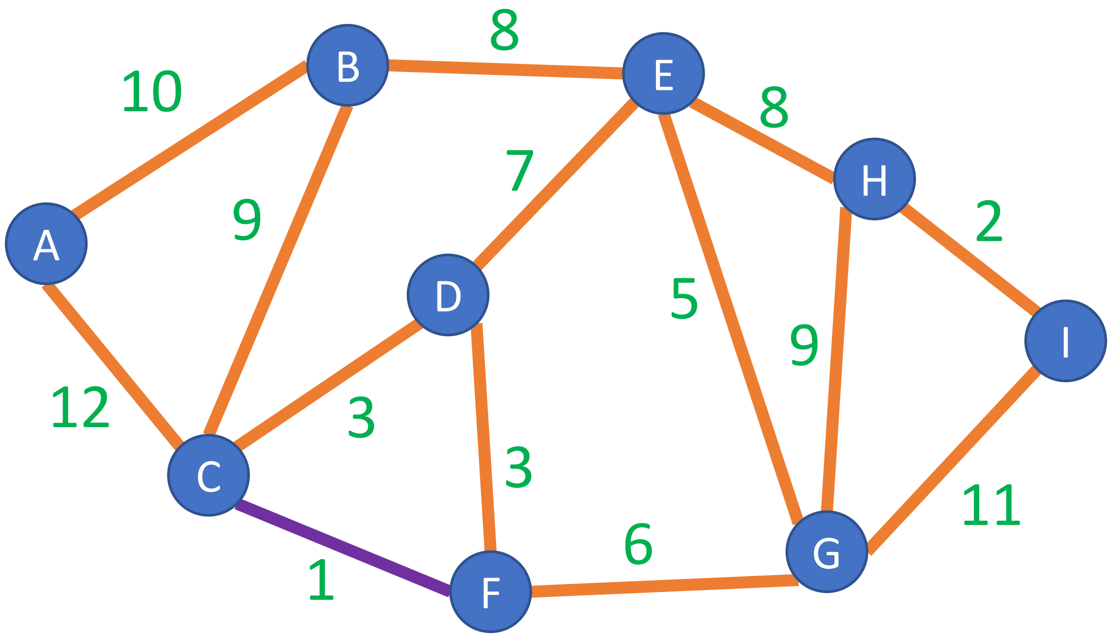

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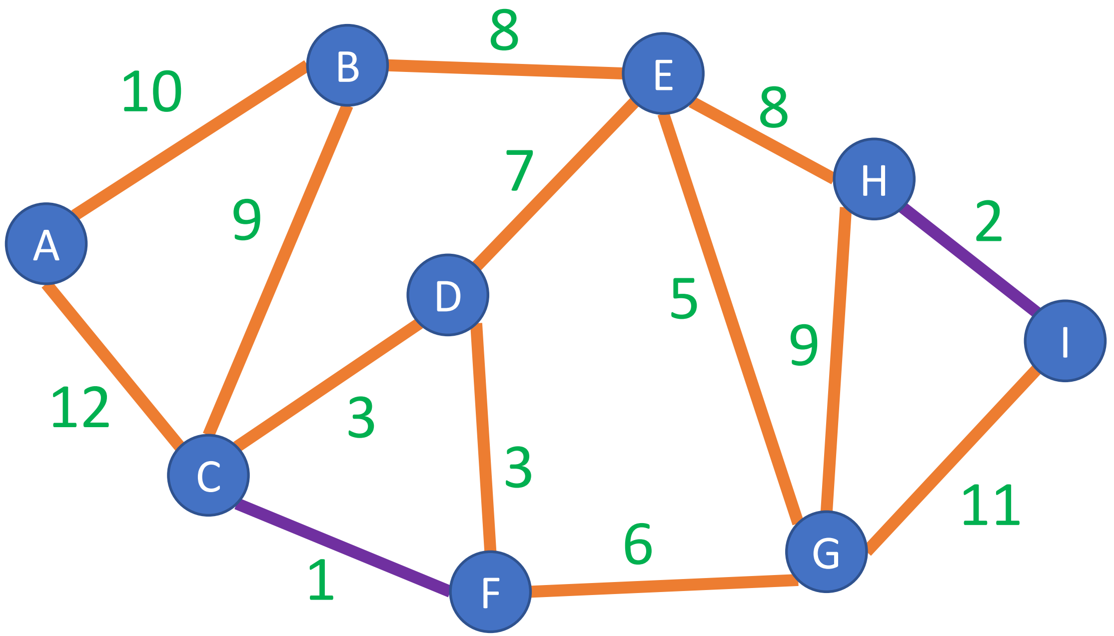

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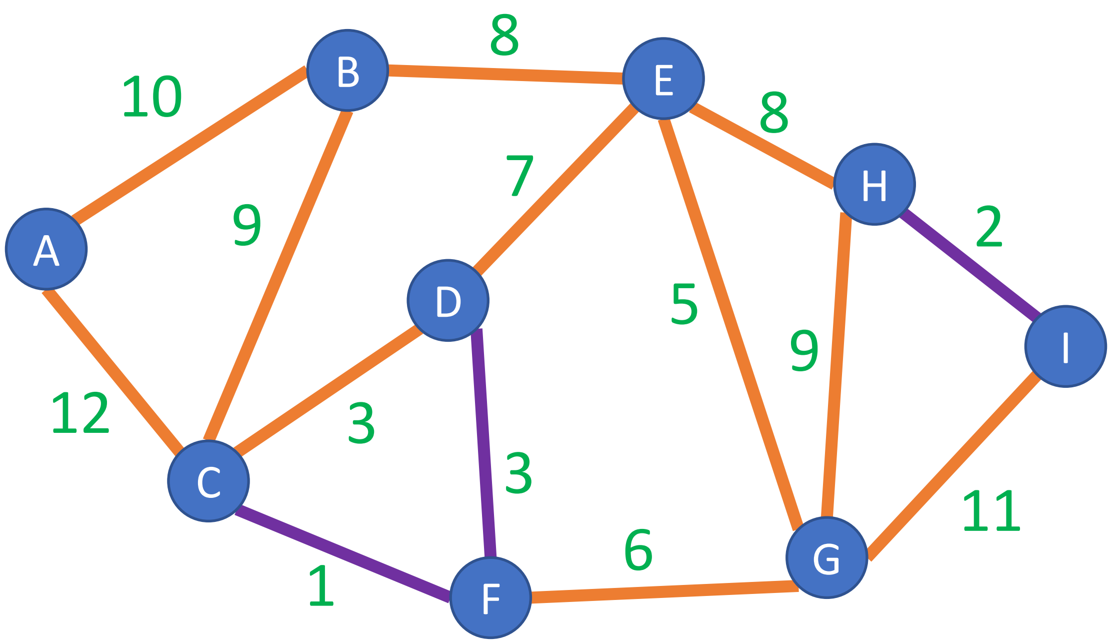

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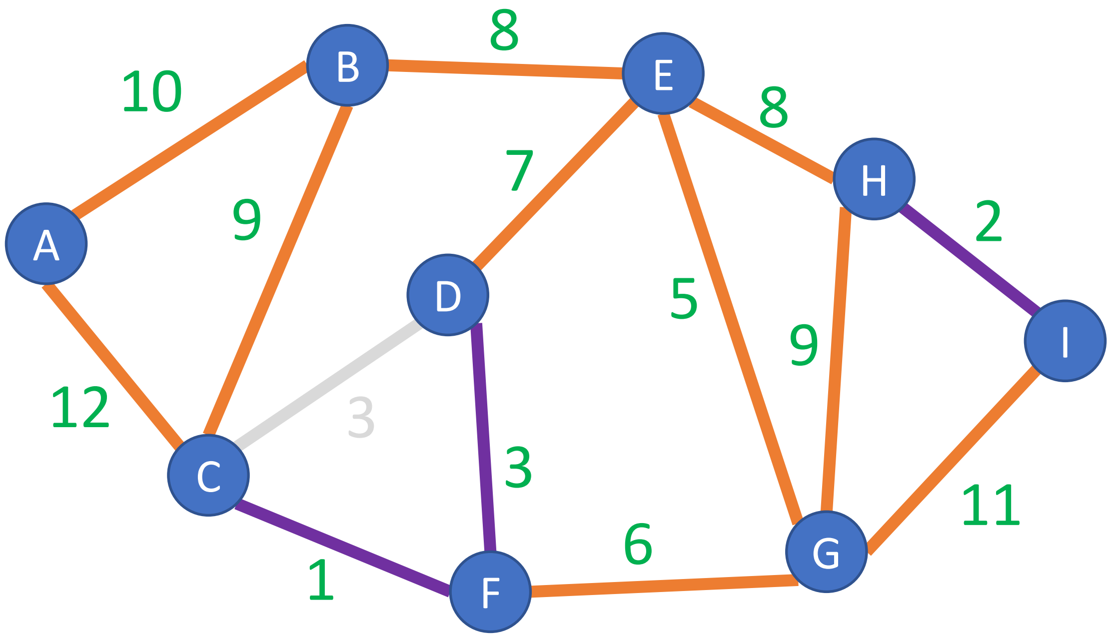

Edge forms a cycle, so do not include

Kruskal's Algorithm

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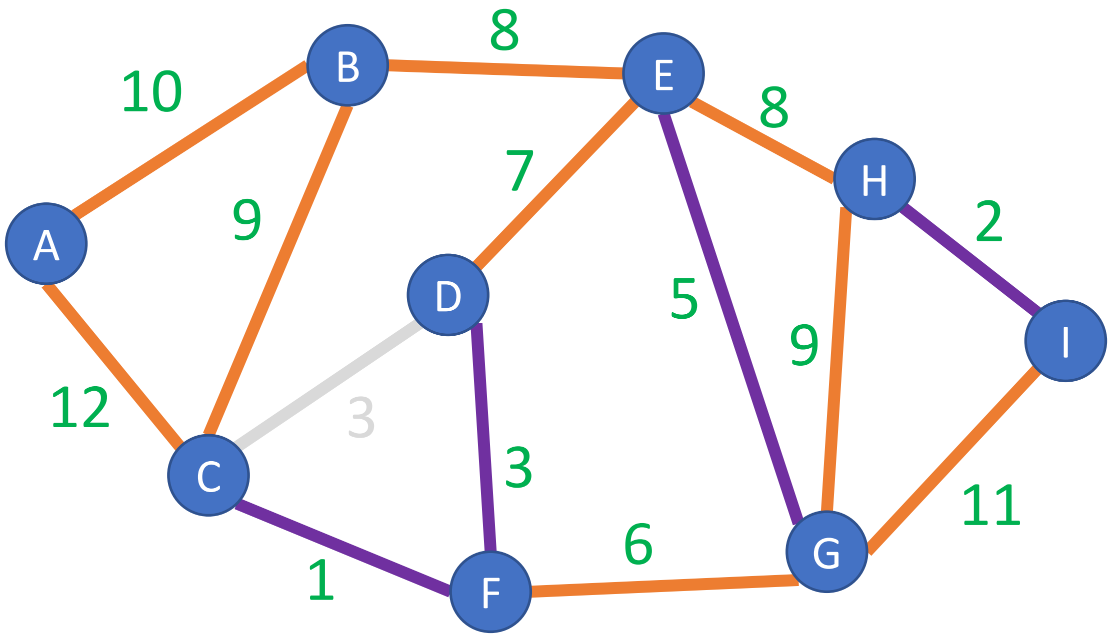

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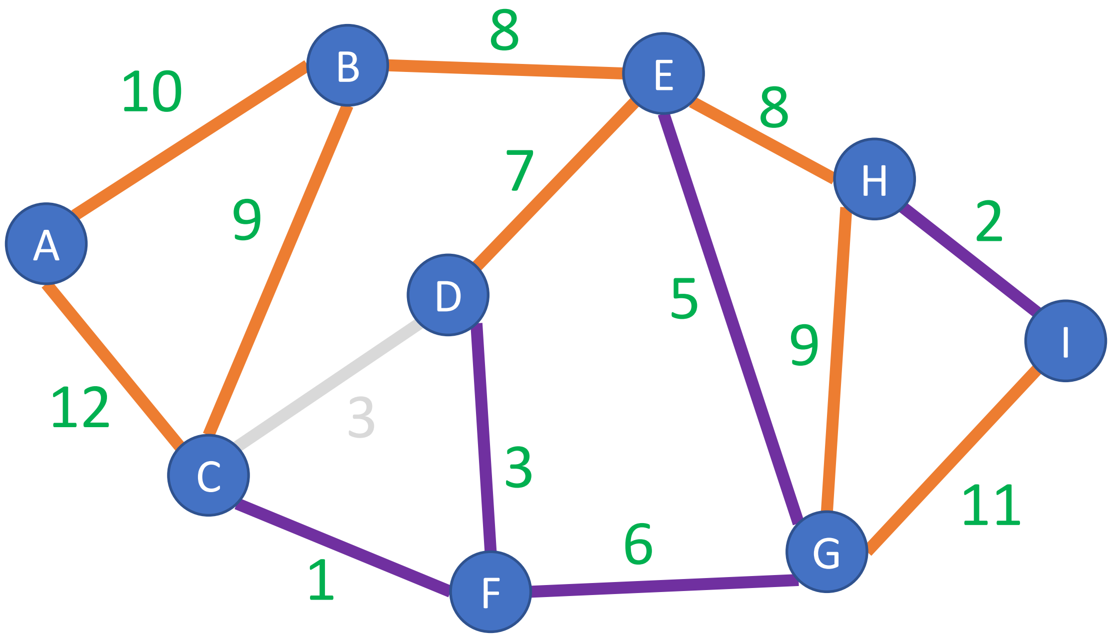

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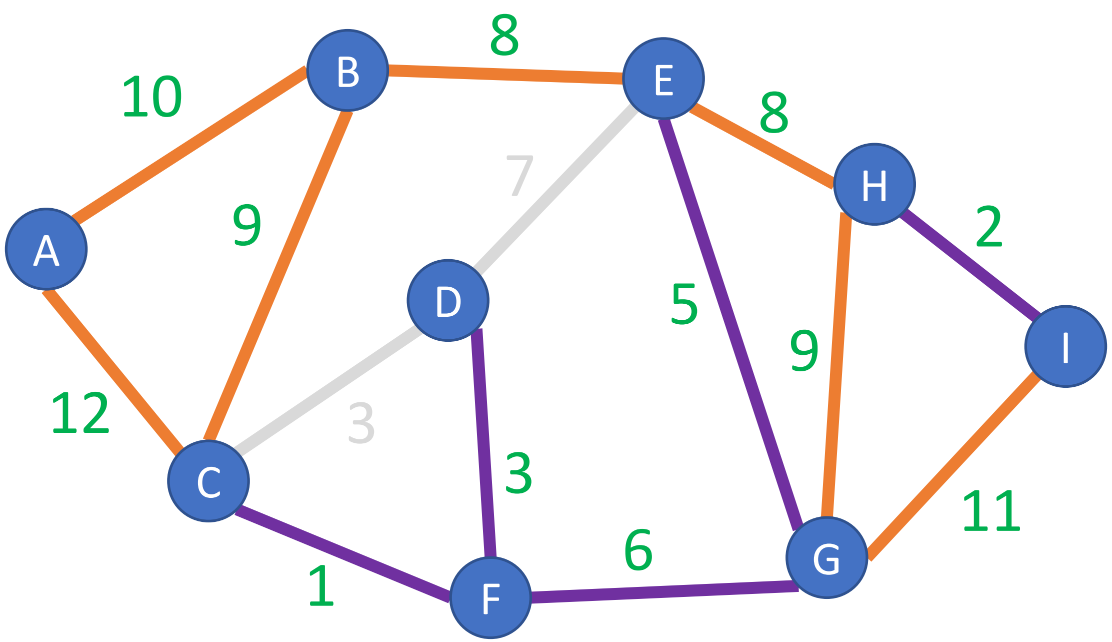

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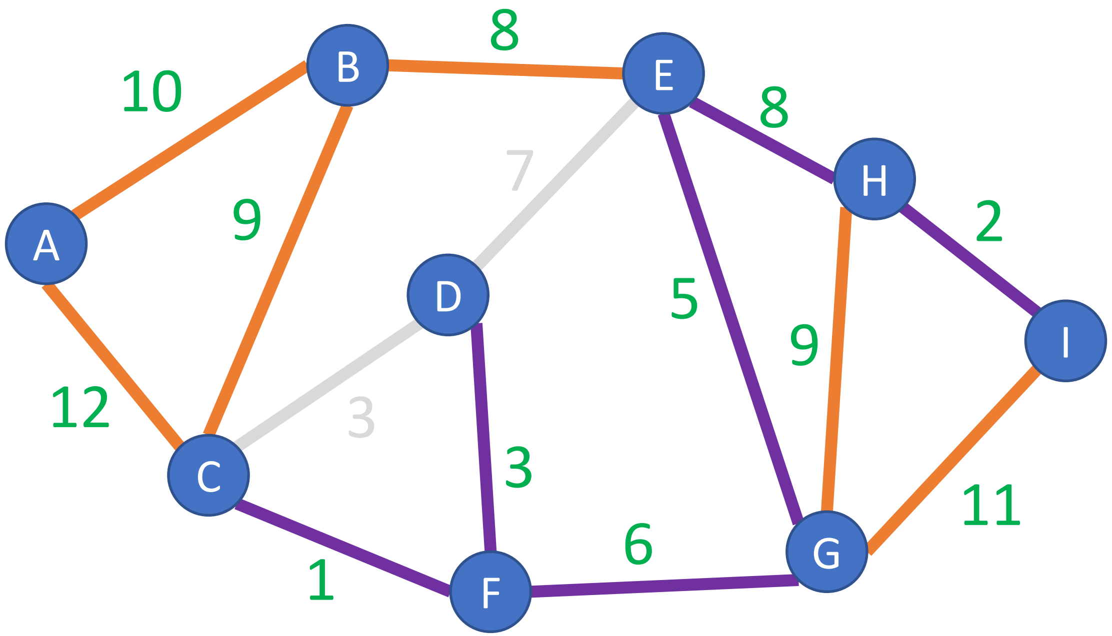

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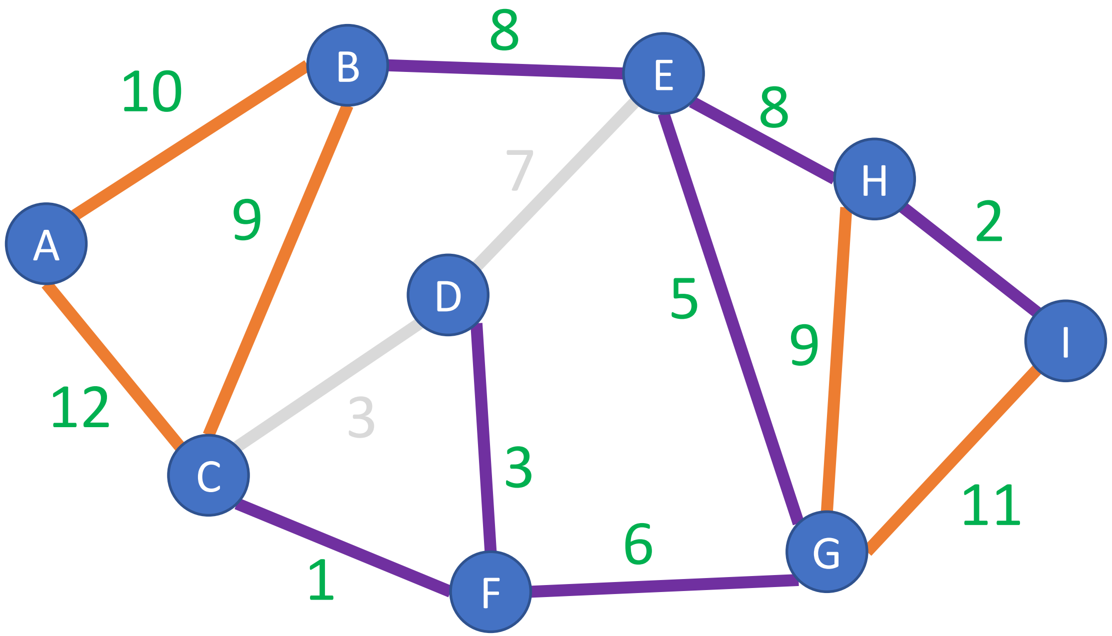

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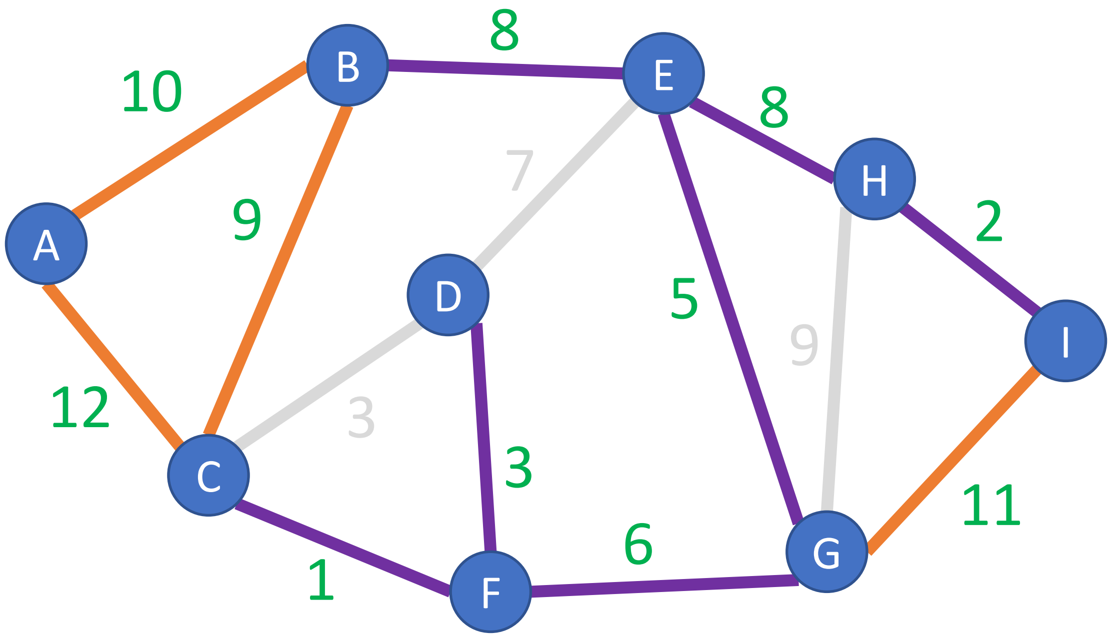

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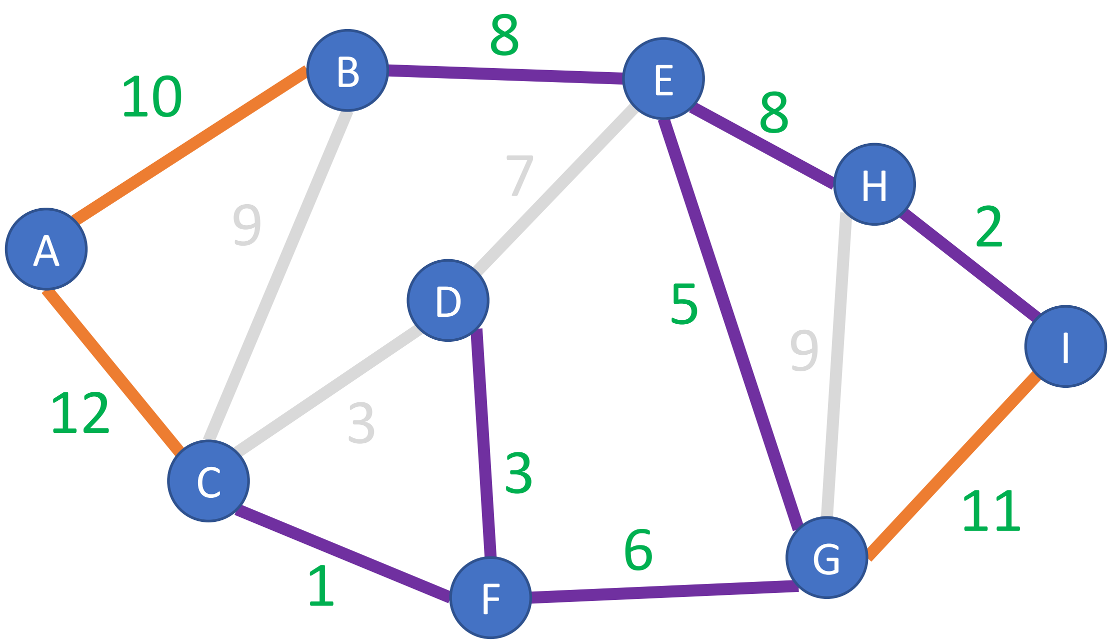
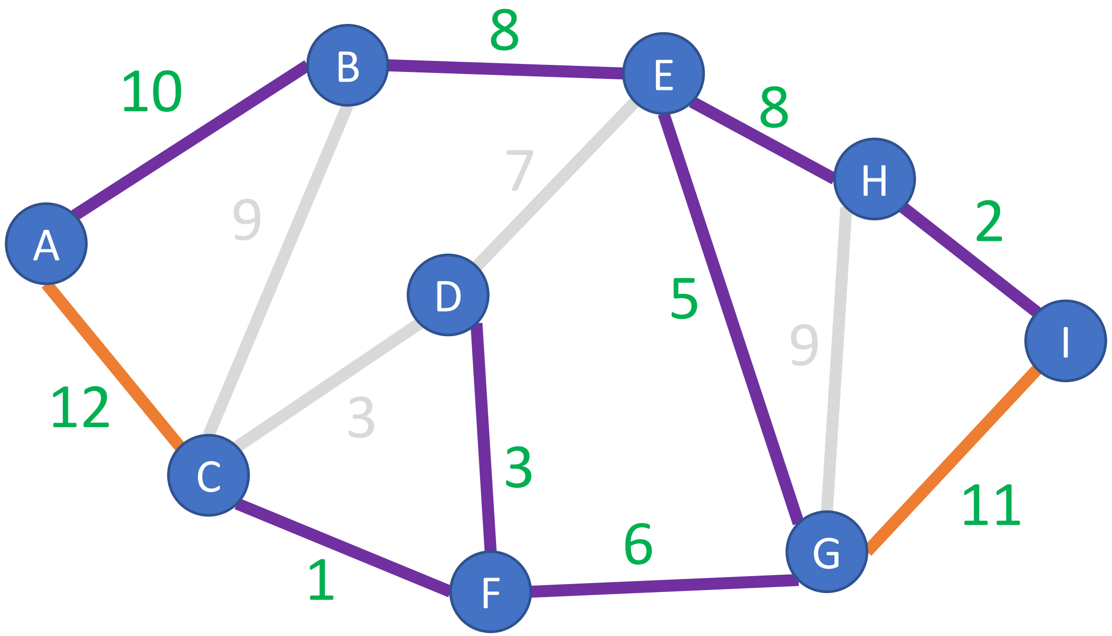


Edge forms a cycle, so do not include

Kruskal's Algorithm

1. Start with an empty set of edges T
2. Repeatedly add to T the lowest-weight edge that does not create a cycle. (Stop when we've added $n-1$ edges.)

$|V|-1$



Now $n-1$ edges have been added.
All nodes are connected.
Algorithm is done!

Kruskal's Algorithm

1. Start with an empty tree T
2. Repeatedly add to T the lowest-weight edge that does not create a cycle

Implementation: iterate over each of the edges in the graph (sorted by weight), and maintain nodes in a union-find (also called disjoint-set) data structure:

- Data structure that tracks elements partitioned into different sets
- **Union:** Merges two sets into one
- **Find:** Given an element, return the index of the set it belongs to
- Both "union" and "find" operations are very fast

Time complexity: $O\big(\alpha(n)\big)$,
where α is the "inverse Ackermann function" (extremely slow-growing function)
for all "practical" n, $\alpha(n) < 5$ (e.g., for all $n < 2^{2^{2^{65536}}} - 3$)

Time Complexity: Kruskal's Algorithm

1. Start with an empty tree T
2. Repeatedly add to T the lowest-weight edge that does not create a cycle

Implementation: iterate over each of the edges in the graph (sorted by weight), and maintain nodes in a union-find (also called disjoint-set) data structure:

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- Both "union" and "find" operations are very fast

- **Overall running time:** $O(|E| \log |E|) = O(|E| \log |V|)$

$$|E| \leq |V|^2 \Rightarrow \log|E| = O(\log|V|)$$

More on Implementation for Kruskal’s

Let *EL* be the set of edges sorted ascending by weight

Consider each vertex to be in a tree of size 1

For each edge *e* in *EL*

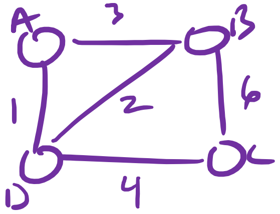
    T1 = tree ID for vertex *head(e)*

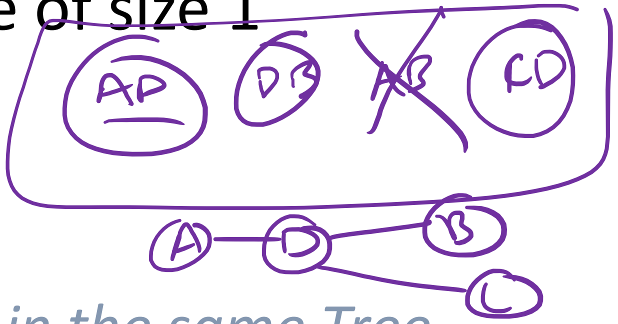
    T2 = tree ID for vertex *tail(e)*

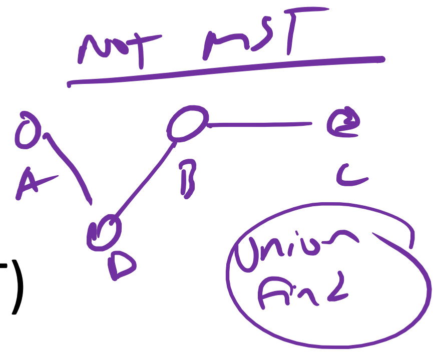
    if (*T1* != *T2*)   *// the nodes are not in the same Tree*

        Add *e* to the output set of edges *T* (which becomes the MST)

        Combine trees *T1* and *T2*







Seems simple, no?

- But, how do you keep track of what tree a vertex is in?
- Trees are sets of vertices. Need to findset(v) and “union” two sets

Union/Find and Disjoint Sets

An Abstract Data Type (ADT) for a collection of sets of any kind of item, where an item can only belong to one of the sets

- We'll assume each item is identified by a unique integer value

Need to support the following operations

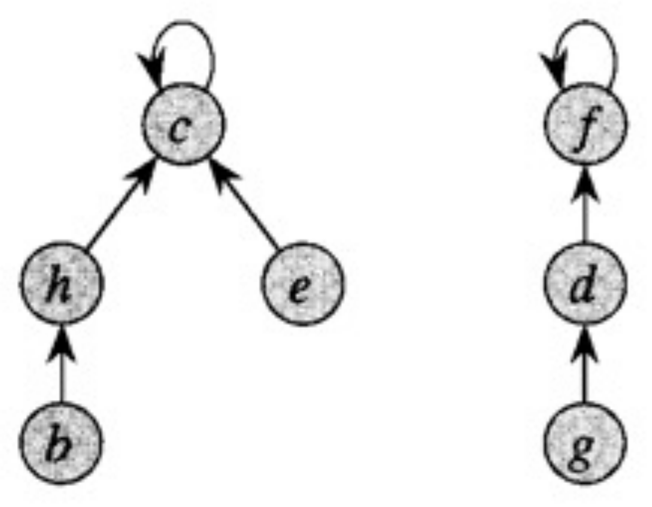
- void makeSet(int n) // construct n independent sets
- int findSet(int i) // given i, which set does i belong to?
- void union(int i, int j) // merge sets containing i and j

Represent Sets As Trees

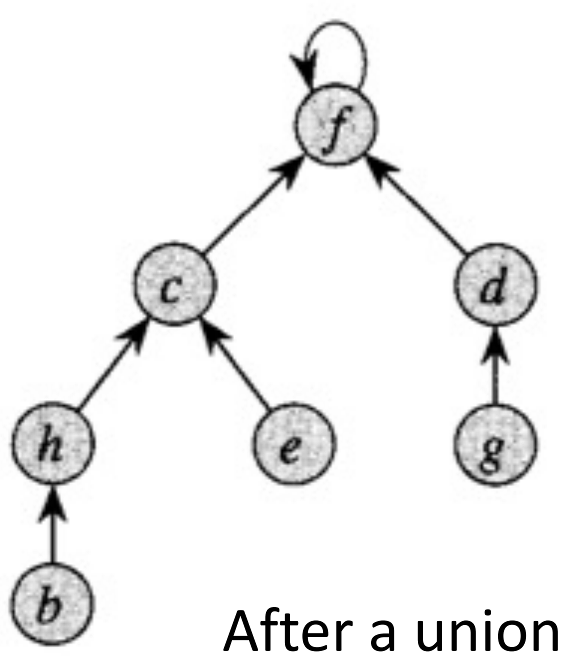
In our implementation, we'll represent each set as a tree

Identify set by its root node's ID (its "label")

- findSet() means tracing up to root
- union() makes one root child of the other root



Two sets



After a union

Union/Find and Disjoint Sets

Needs to support the following operations

- void makeSet(int n)   //construct n independent sets

Solution:

- Store as array of size n. Each location stores label for that set.

0	1	2	3	4	5	6	7
0	*1*	*2*	*3*	*4*	*5*	*6*	*7*

Union/Find and Disjoint Sets

Needs to support the following operations

- int findSet(int i) //given i, which set does i belong to?

Solution: Trace around array until we find place where index and contents match

- Start at index i and repeat:
  - If a[i] == i then return i
  - Else set i = a[i]

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Union/Find and Disjoint Sets

Needs to support the following operations

- void union(int i, int j)   //merge sets i and j

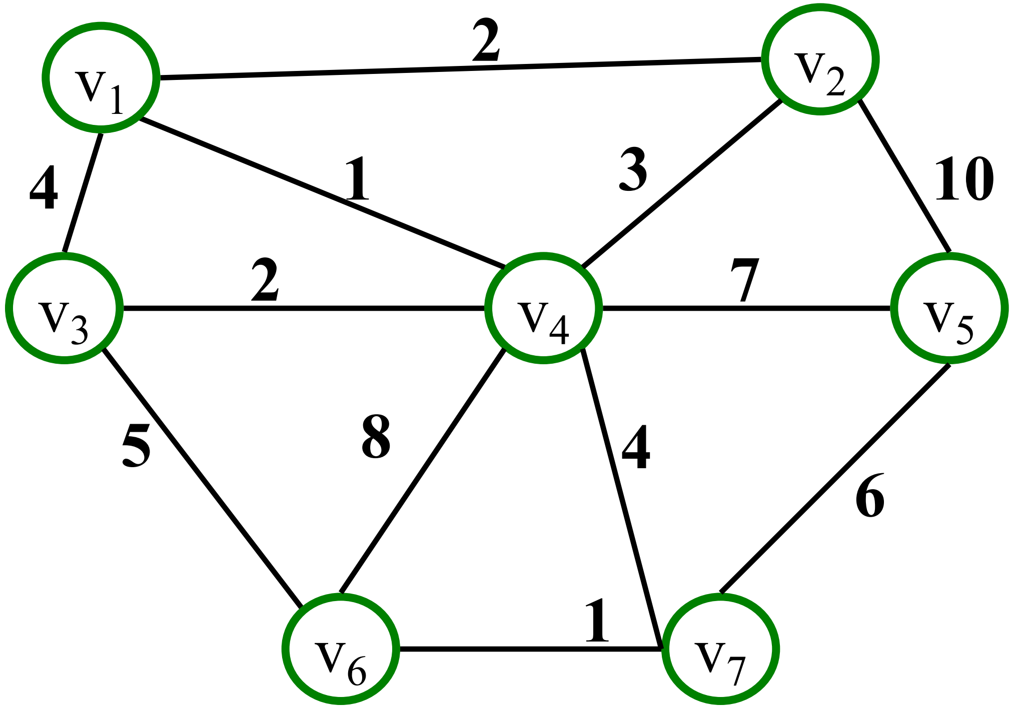
Solution: find label for each set (call find() method), then set one label to point to other

- Label1 = find(i); Label2 = find(j)
- a[Label1] = Label2 //OR a[Label2] = Label1

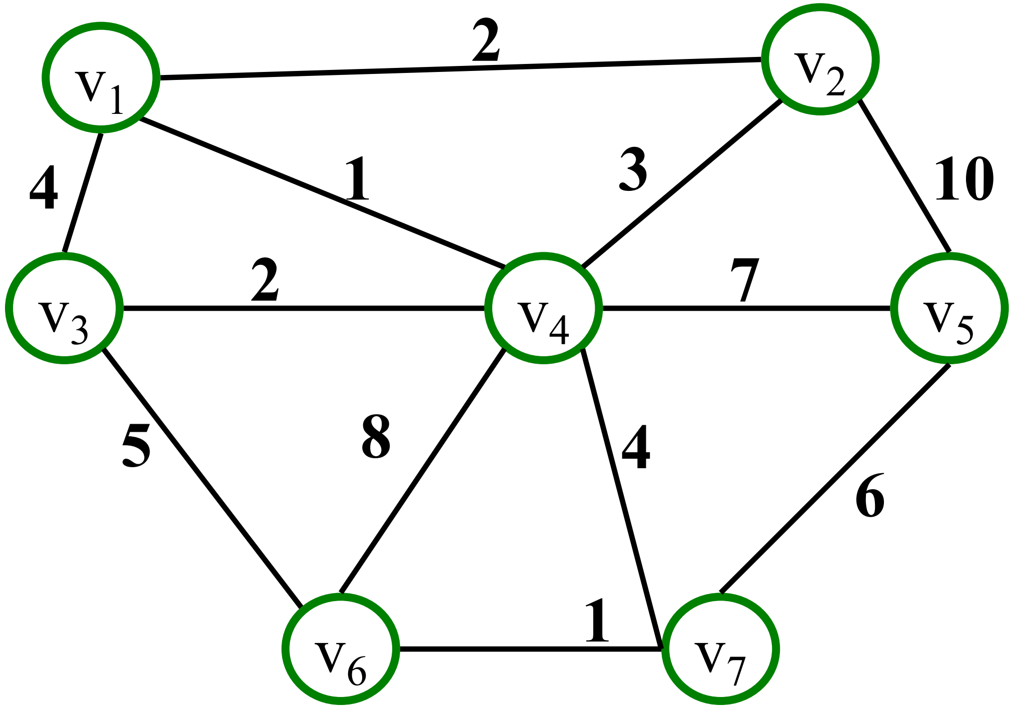
0	1	3	3	5	6	1	~~7~~ 4
0	*1*	*2*	*3*	*4*	*5*	*6*	*7*

Practice

Can you do Prim's MST on This?



MST



v1

{v1, v4}

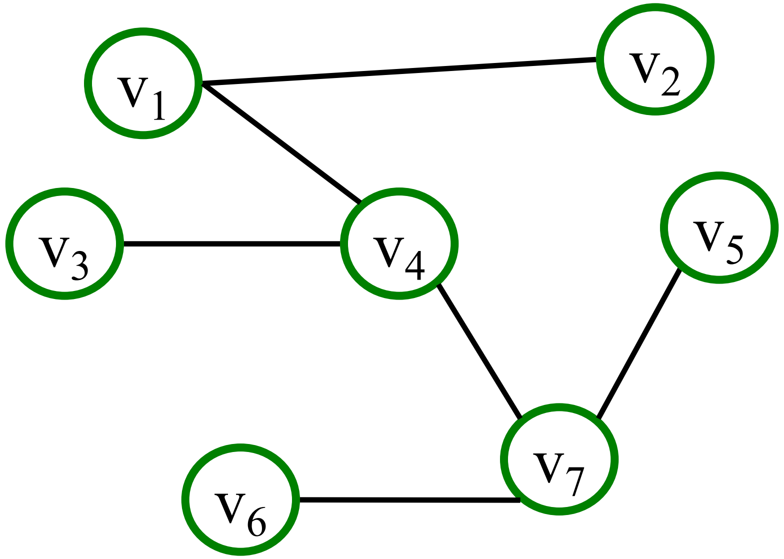
{v1, v2}

{v4, v3}

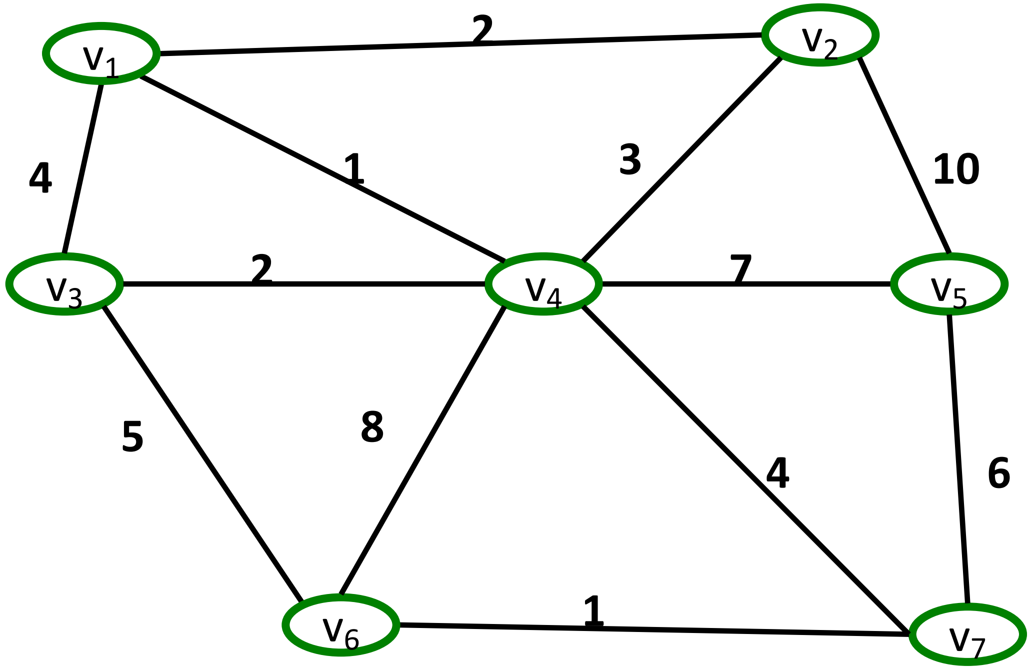
{v4, v7}

{v7, v6}

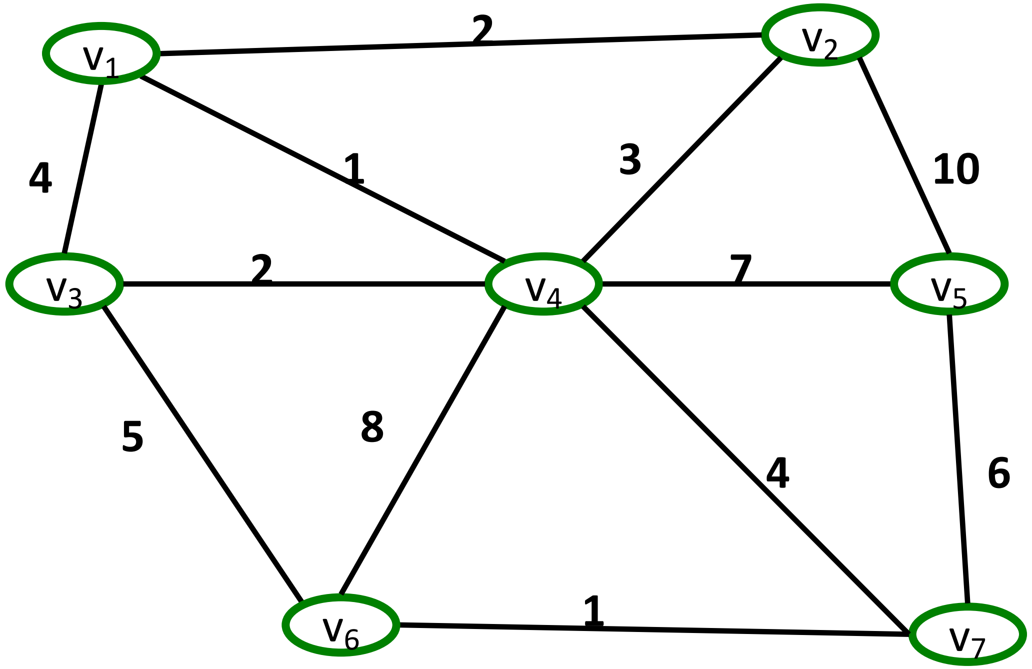
{v7, v5}

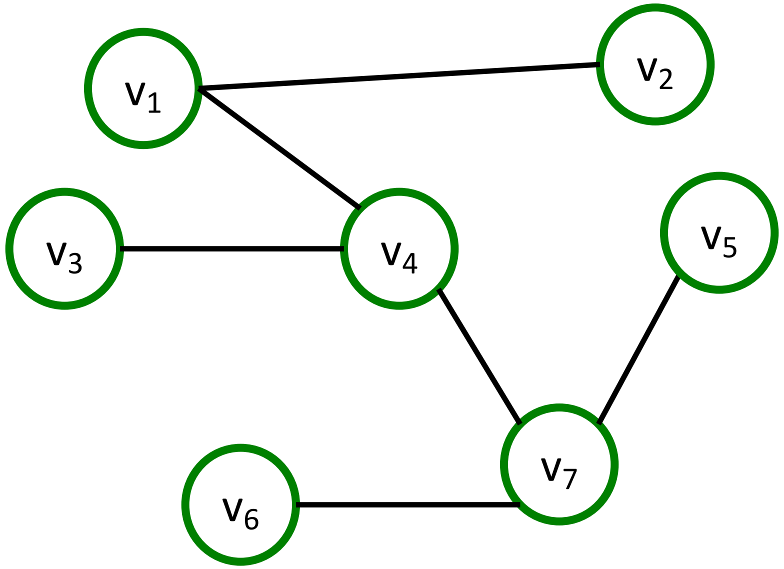


Can you do Kruskal's MST on This?



MST and Kruskal's Example





Cost(MST) = 16

Disjoint Sets and Find/Union Algorithms

Readings: CLRS 19.3

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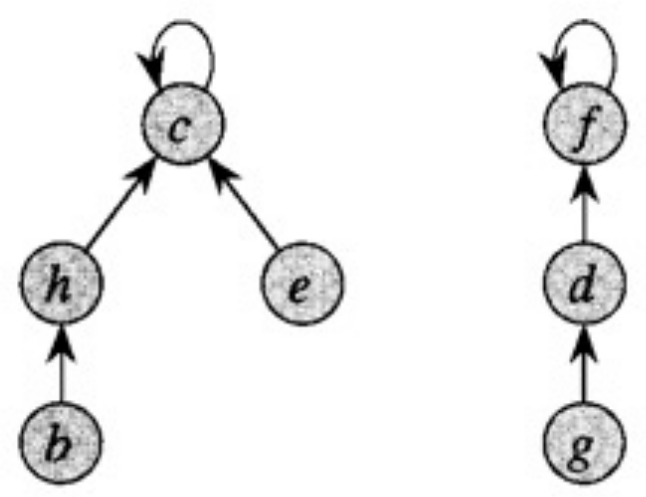
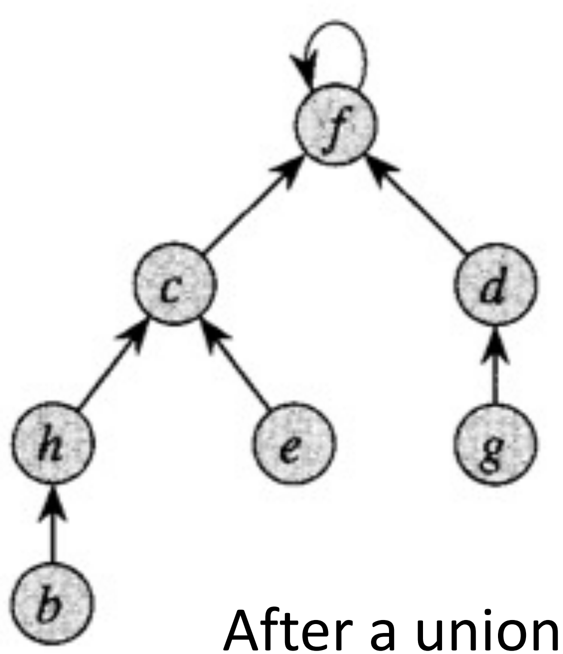


Two sets



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- Store as array of size n. Each location stores label for that set.

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Union/Find and Disjoint Sets

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- void union(int i, int j)   //merge sets i and j

Solution: find label for each set (call find() method), then set one label to point to other

- Label1 = find(i); Label2 = find(j)
- a[Label1] = Label2 //OR a[Label2] = Label1

0	1	3	3	5	6	1	7
0	*1*	*2*	*3*	*4*	*5*	*6*	*7*

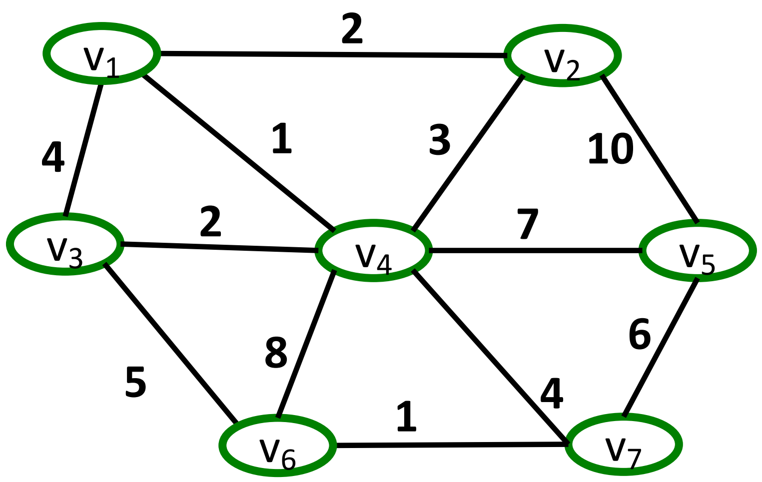
Union/Find and Disjoint Sets

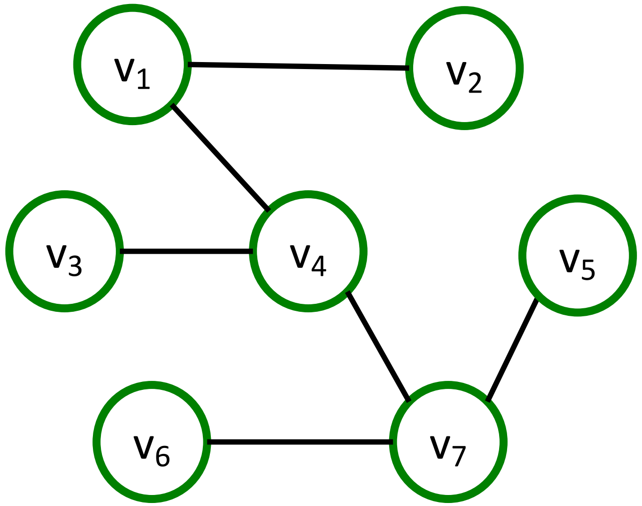
Example:

- union(4,5)
- union(6,7)
- union(1,2)
- union(5,6)
- find(1); find(4); find(6)

0	1	2	3	4	5	6	7
0	*1*	*2*	*3*	*4*	*5*	*6*	*7*

Example Using MST Example





Union/Find and Disjoint Sets

Time-complexity, where n is size of array?

makeSet()

- Linear: just create array and fill it with values

find()

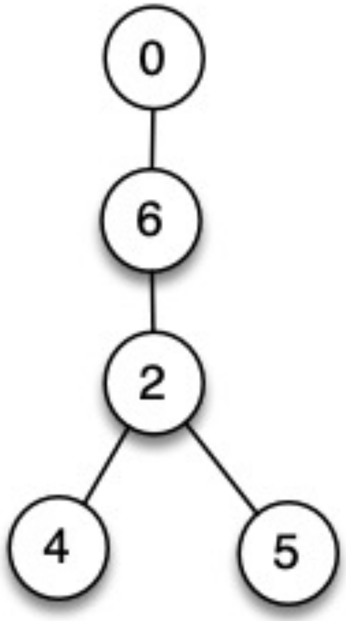
- Linear if have to trace a long way to get to label
- Constant if lucky and input is the label (root note) or near it

union()

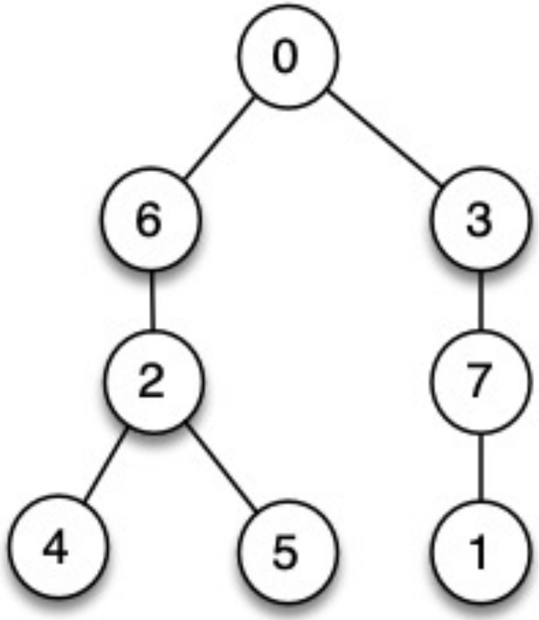
- Constant to change the label BUT…
- Could be linear to find the two labels first.

Optimization 1: Union by rank

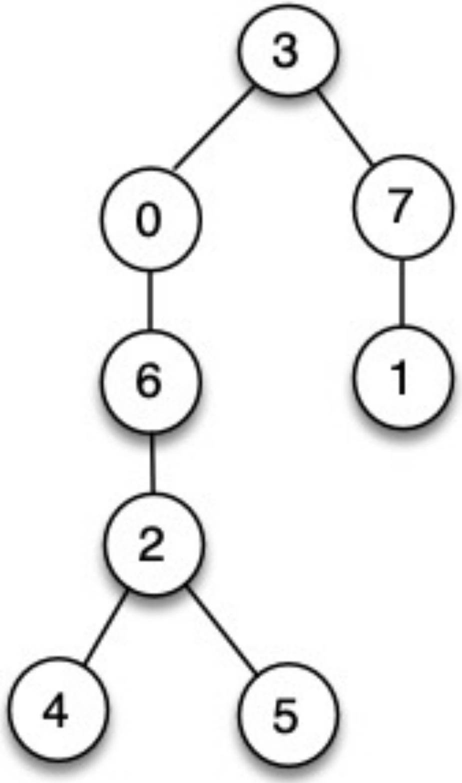
Two Sets:



Union'd under 0:



Union'd under 3:



Optimization 1: Union by rank

Easy to implement!!

What's "rank" here?

- Upper bound on height of a node in our set's tree

Union by rank:

- Make the root with smaller rank point to the root with larger rank

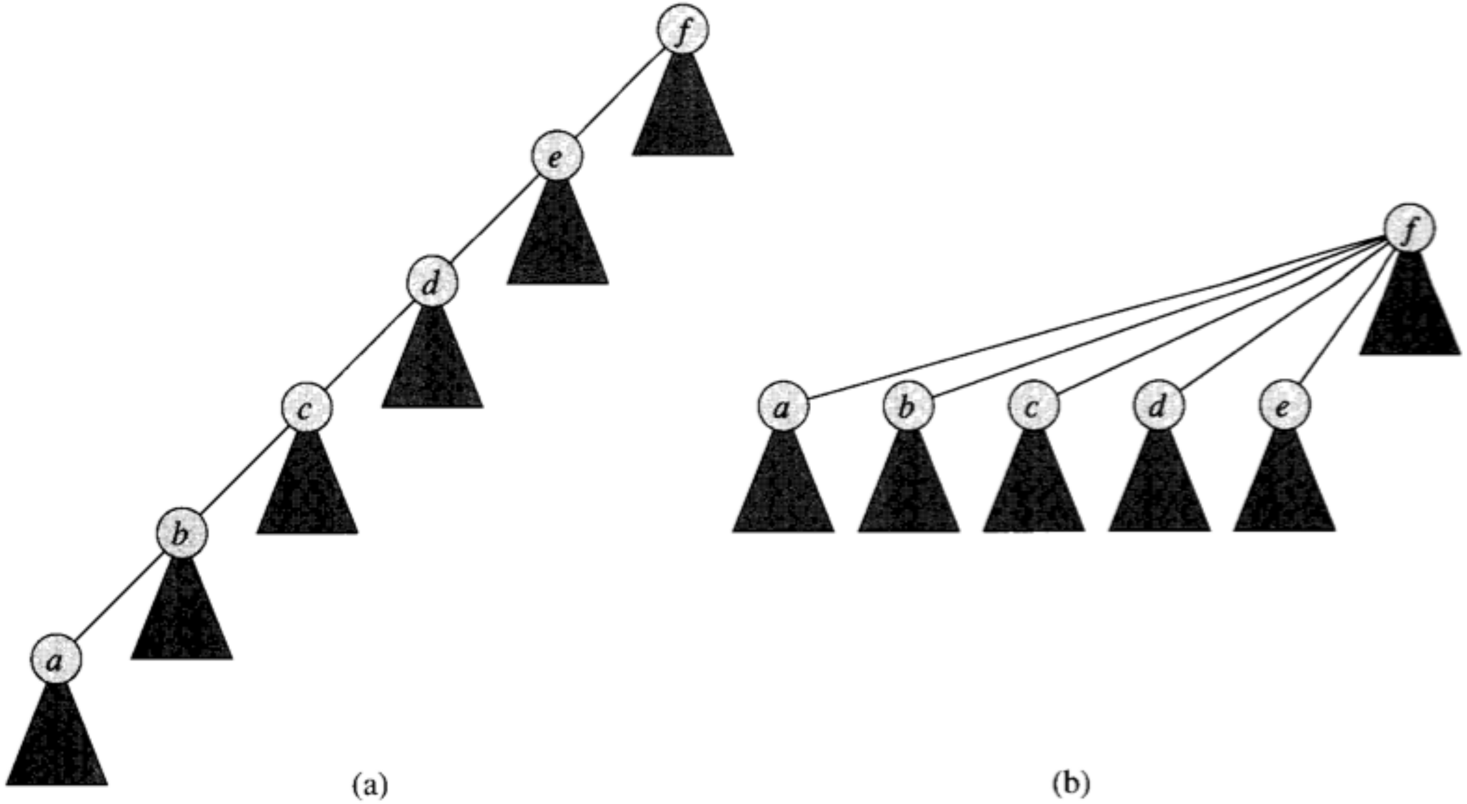
```
MAKE-SET(x)
    x.p = x
    x.rank = 0

UNION(x, y)
    LINK(FIND-SET(x), FIND-SET(y))

LINK(x, y)
    if x.rank > y.rank
        y.p = x
    else x.p = y
        if x.rank == y.rank
            y.rank = y.rank + 1
```

Optimization 2: Path Compression

Nothing special about tree's structure, as long as we can trace back to root

Idea: as we do a find, each node we visit gets updated to point directly to root

Later finds will be faster



Optimization 2: Path Compression

Also easy to implement

- CLRS code uses recursion →
- Or would loop and keep a list

```
def find_set(x):
  path = []
  while x != x.p:
     path.append(x)
     x = x.p
  for n in path:
     n.p = x.p
  return x.p
```

FIND-SET(x)

if $x \neq x.p$
    $x.p =$ FIND-SET$(x.p)$
return $x.p$

Complexity for Kruskal's

Union-by-rank and path compression yields m operations in $\Theta\left(m * \alpha(n)\right)$

- where $\alpha(n)$ a VERY slowly growing function. (See textbook for details)
- m is the number of times you run the operation. So constant time, for each operation

So overall Kruskal's with path compression:

$\Theta(E * \log(V) + E * 1) = \Theta(E * \log(V))$ //now the heap is slowest part

Originally:

$\Theta(E * \log(V) + E * V) = \Theta(E * V) = \boldsymbol{O}\left(\boldsymbol{V^3}\right)$ *//Assumed find and union linear time*

Summary

What did we learn?

Minimum Spanning Trees

Prim’s Algorithm

- Very similar to Dijkstra’s SP algorithm
- Different greedy choice to add next edge to tree

Kruskal’s Algorithm

Find-union

- How to implement
- How to optimize
- How it affects runtime of Kruskal’s algorithm.