CS 3100
Data Structures and Algorithms 2
Lecture 10: D&C: CPP & Matrix Multiply

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Spring 2024

Readings in CLRS 4th edition:

- Section 4.5

Announcements

- PS4 due tomorrow
- PA2 due next Friday, March 1, 2024
- Quizzes 1-2 coming February 29, 2024
  - Both quizzes taken the same day
  - If you have SDAC, please schedule for 1 exam (*not a quiz*)
- Office hours
  - Prof Hott Office Hours: Mondays 11a-12p, Fridays 10-11a and 2-3p
  - Prof Pettit Office Hours: Mondays and Wednesdays 2:30-4:00p
  - TA office hours posted on our website
  - Office hours are not for "checking solutions"

Divide and Conquer

[CLRS Chapter 4]

Divide:

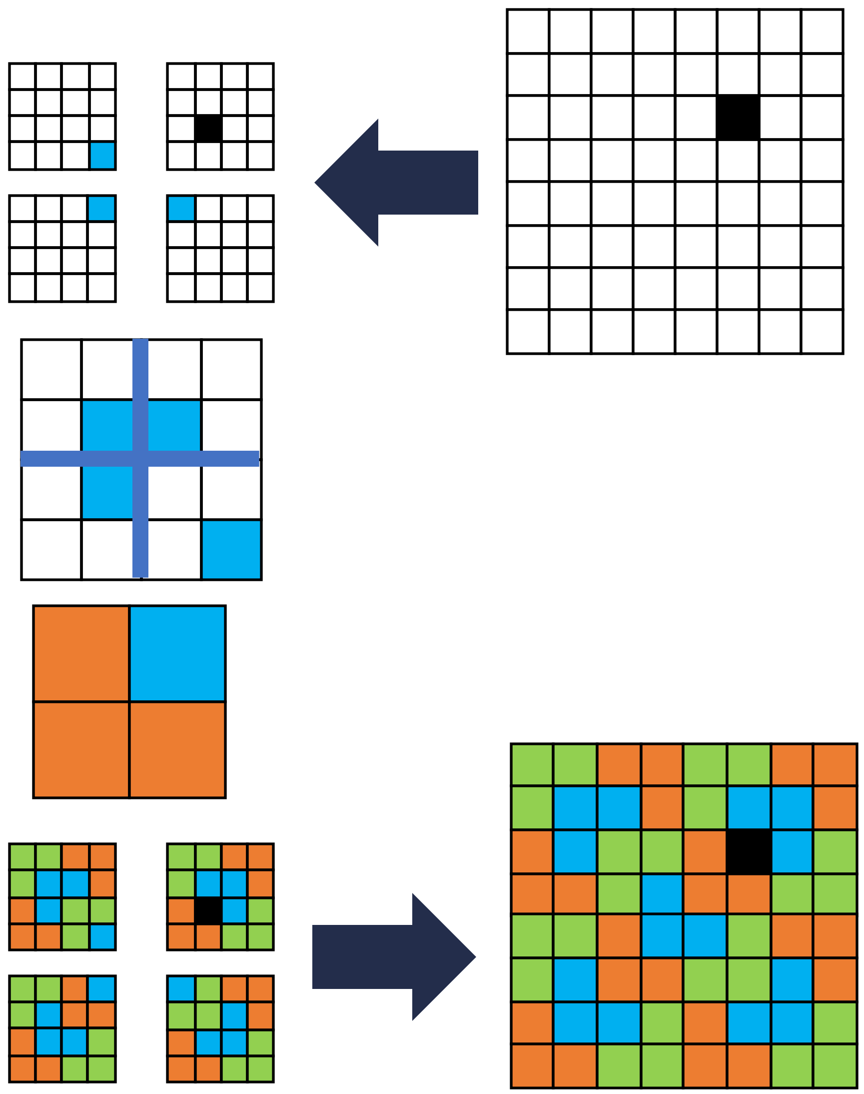
- Break the problem into multiple subproblems, each smaller instances of the original

Conquer:

- If the suproblems are "large":
  - Solve each subproblem recursively
- If the subproblems are "small":
  - Solve them directly (base case)

Combine:

- Merge solutions to subproblems to obtain solution for original problem

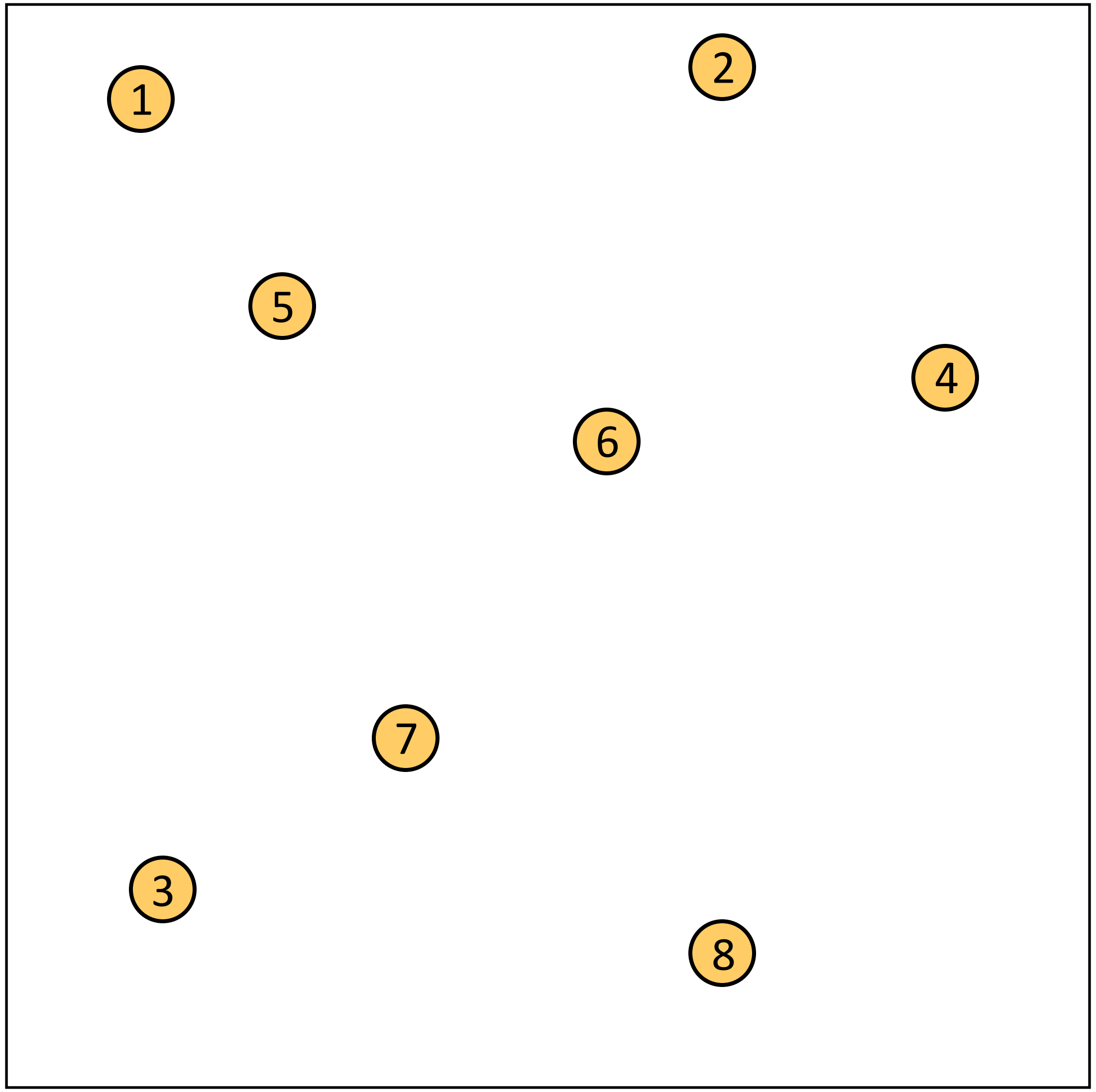


Closest Pair of Points

Given: A list of points

Return: Pair of points with smallest distance apart

$\Theta(n \log n)$



Closest Pair of Points: Divide and Conquer

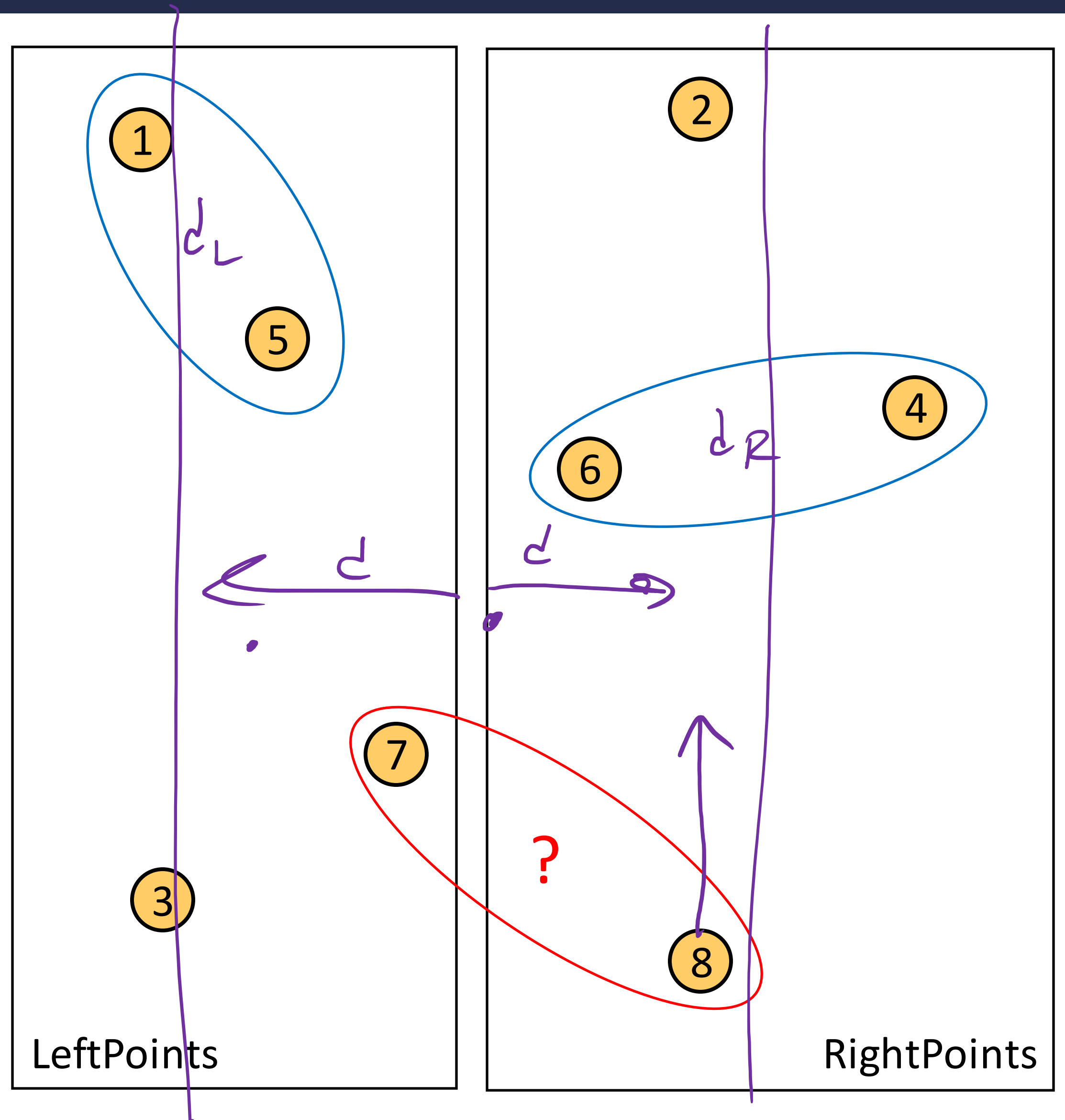
Initialization: Sort points by x-coordinate

Divide: Partition points into two lists of points based on x-coordinate

Conquer: Recursively compute the closest pair of points in each list

Combine:

- Construct list of points in the boundary
- Sort runway points by y-coordinate
- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points



Closest Pair of Points: Divide and Conquer

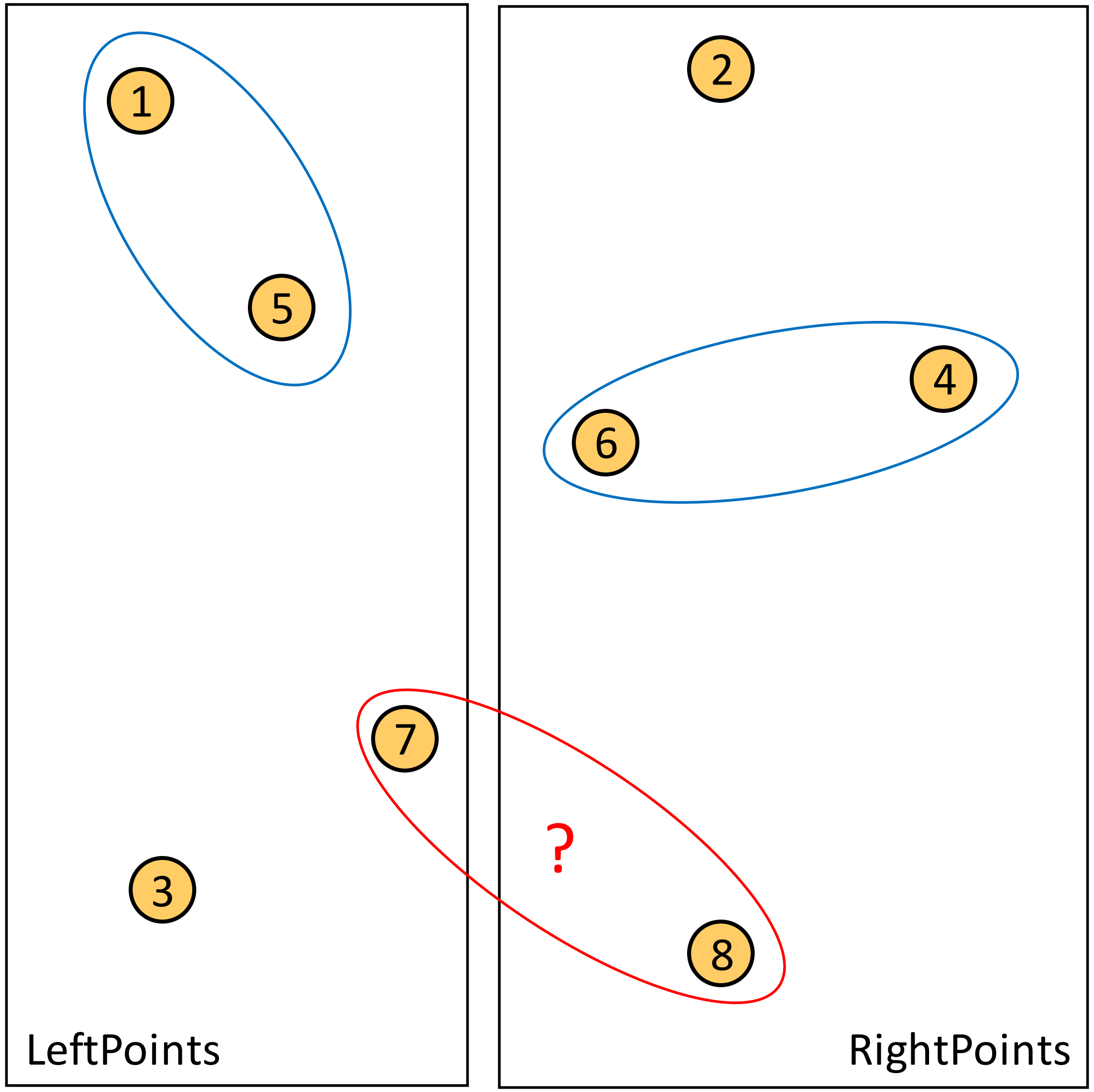
Initialization: Sort points by x-coordinate

Divide: Partition points into two lists of points

Looks like another $O(n \log n)$ algorithm – combine step is still too expensive

Combine:
- Construct list of points in t boundary
- Sort runway points by y-coordinate
- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points $n^{\log_b a} \approx f(n)$

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n) \quad = \Theta(n \log n)$$



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- Construct list of points in the boundary
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- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points

Solution: Maintain additional information in the recursion
- Minimum distance among pairs of points in the list
- List of points sorted according to y-coordinate

Sorting runway points by y-coordinate now becomes a **merge**

Listing Points in the Boundary

LeftPoints:

Closest Pair: $(1,5)$, $d_{1,5}$

Sorted Points: [3,7,5,1]

RightPoints:

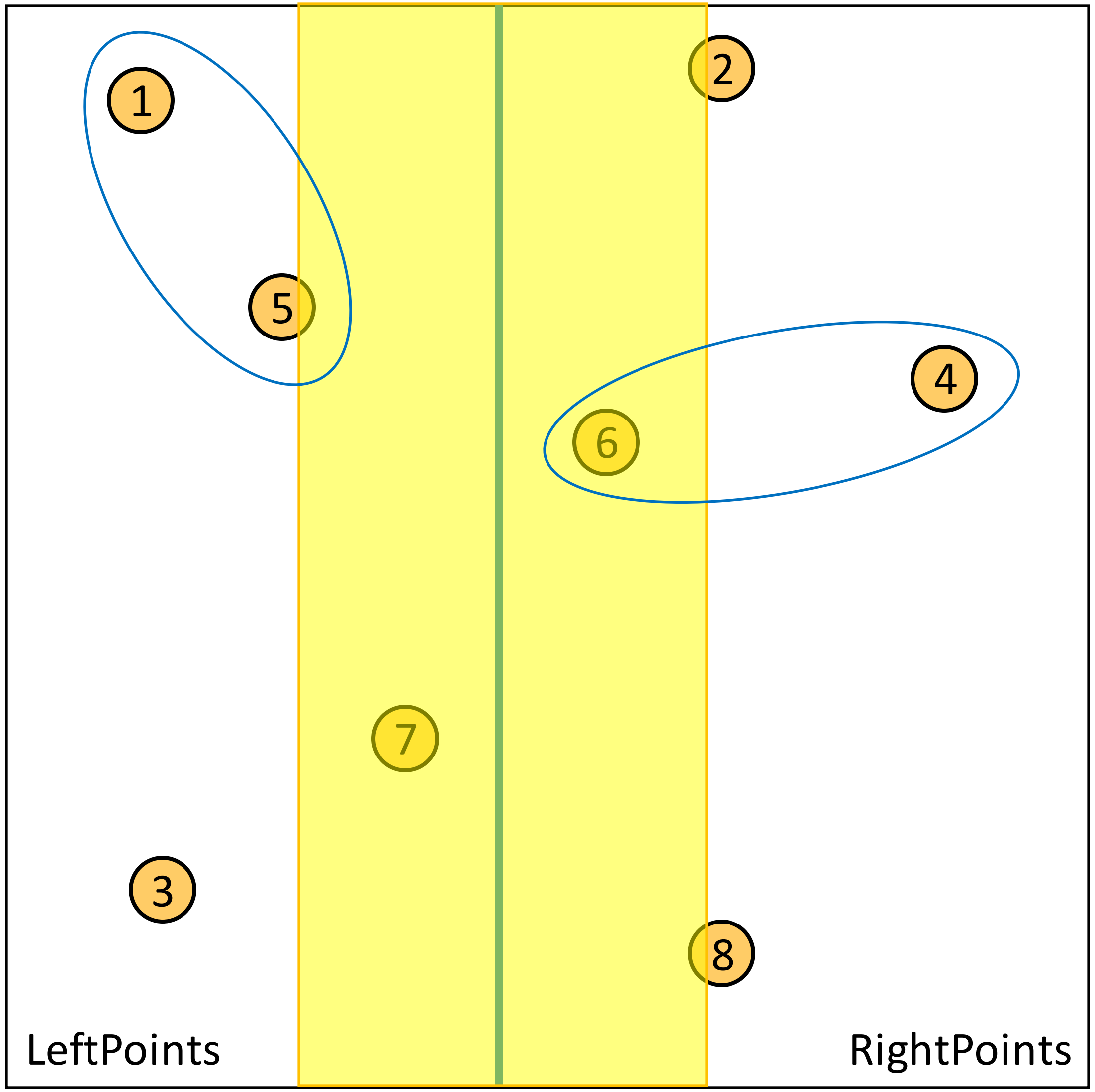
Closest Pair: $(4,6)$, $d_{4,6}$

Sorted Points: [8,6,4,2]

Merged Points: [8,3,7,6,4,5,1,2]

Runway Points: [8,7,6,5,2]

Both of these lists can be computed by a *single* pass over the lists



Closest Pair of Points: Divide and Conquer

Initialization: Sort points by x-coordinate

Divide: Partition points into two lists of points based on x-coordinate

Conquer: Recursively compute the closest pair of points in each list

Combine:

- Construct list of points in the boundary
- Sort runway points by y-coordinate
- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points

Initialization: Sort points by x-coordinate

Divide: Partition points into two lists of points based on x-coordinate

Conquer: Recursively compute the closest pair of points in each list and each list sorted by y-coordinate

Combine:

- Merge sorted list of points by y-coordinate and construct list of points in the runway (sorted by y-coordinate)
- Compare each point in runway to 15 points above it and save the closest pair
- Output closest pair among left, right, and runway points and y-sorted list

Closest Pair of Points: Divide and Conquer

What is the running time?

$\Theta(n \log n)$

$T(n)$

$a=2,\ b=2 \quad n^{\log_b a} = n^{\log_2 2} = n$

$$T(n) = 2T(n/2) + \Theta(n)$$

Case 2 of Master's Theorem:

$$T(n) = \Theta(n \log n)$$

$\Theta(n \log n)$	**Initialization:** Sort points by x-coordinate
$\Theta(1)$	**Divide:** Partition points into two lists of points based on x-coordinate
$2T(n/2)$	**Conquer:** Recursively compute the closest pair of points in each list and each list sorted by y-coordinate
	Combine:
$\Theta(n)$	• Merge sorted list of points by y-coordinate and construct list of points in the runway (sorted by y-coordinate)
$\Theta(n)$	• Compare each point in runway to 15 points above it and save the closest pair
$\Theta(1)$	• Output closest pair among left, right, and runway points and y-sorted list

Matrix Multiplication

$$\begin{array}{cc} & n \\ n & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \end{array} \times \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 2+16+42 & 4+20+48 & 6+24+54 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$\begin{array}{cc} \overbrace{\hphantom{60 \quad 72 \quad 84}}^{n} & \\ = \begin{bmatrix} 60 & 72 & 84 \\ 132 & 162 & 192 \\ 204 & 252 & 300 \end{bmatrix} & n \end{array}$$

$n \times n = n^2$

Run time? $O(n^3)$

Lower Bound? $\Omega(n^2)$

Matrix Multiplication Divide and Conquer

Multiply $n \times n$ matrices (A and B)

Divide:

$$A = \left[\begin{array}{cc|cc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \hline a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{array}\right] \qquad B = \left[\begin{array}{cc|cc} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \\ \hline b_9 & b_{10} & b_{11} & b_{12} \\ b_{13} & b_{14} & b_{15} & b_{16} \end{array}\right]$$

Matrix Multiplication Divide and Conquer

Multiply $n \times n$ matrices (A and B)

$$\begin{array}{ccc} n/2 & n/2 & \\ & & \end{array}$$

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{array}{c} n/2 \\ n/2 \end{array} \qquad B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

Combine:

$$AB = \begin{bmatrix} A_{1,1}B_{1,1} + A_{1,2}B_{2,1} & A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ A_{2,1}B_{1,1} + A_{2,2}B_{2,1} & A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{bmatrix}$$

Run time? $T(n) = 8T\left(\frac{n}{2}\right) + \boxed{4\left(\frac{n}{2}\right)^2}$ Cost of additions

Matrix Multiplication Divide and Conquer

$$T(n) = 8T\left(\frac{n}{2}\right) + 4\left(\frac{n}{2}\right)^2$$

$$T(n) = 8T\left(\frac{n}{2}\right) + n^2$$

$$a = 8, b = 2, f(n) = n^2$$

Case 1!

$$n^{\log_b a} = n^{\log_2 8} = n^3$$

$$T(n) = \Theta(n^3)$$

Can we do better?

Matrix Multiplication Divide and Conquer

Multiply $n \times n$ matrices (A and B)

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \qquad B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

$$AB = \boxed{\begin{bmatrix} A_{1,1}B_{1,1} + A_{1,2}B_{2,1} & A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ A_{2,1}B_{1,1} + A_{2,2}B_{2,1} & A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{bmatrix}}$$

Idea: Use a Karatsuba-like technique on this

Strassen’s Algorithm

Multiply $n \times n$ matrices (A and B)

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \qquad B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

Calculate:

$$Q_1 = (A_{1,1} + A_{2,2})(B_{1,1} + B_{2,2})$$
$$Q_2 = (A_{2,1} + A_{2,2})B_{1,1}$$
$$Q_3 = A_{1,1}(B_{1,2} - B_{2,2})$$
$$Q_4 = A_{2,2}(B_{2,1} - B_{1,1})$$
$$Q_5 = (A_{1,1} + A_{1,2})B_{2,2}$$
$$Q_6 = (A_{2,1} - A_{1,1})(B_{1,1} + B_{1,2})$$
$$Q_7 = (A_{1,2} - A_{2,2})(B_{2,1} + B_{2,2})$$

Find AB:

$$AB = \begin{bmatrix} Q_1 + Q_4 - Q_5 + Q_7 & Q_3 + Q_5 \\ Q_2 + Q_4 & Q_1 - Q_2 + Q_3 + Q_6 \end{bmatrix}$$

7 Multiplications

18 Additions

$$T(n) = 7T\left(\frac{n}{2}\right) + 18\frac{n^2}{4}$$

Strassen's Algorithm

$$T(n) = 7T\left(\frac{n}{2}\right) + \frac{9}{2}n^2$$

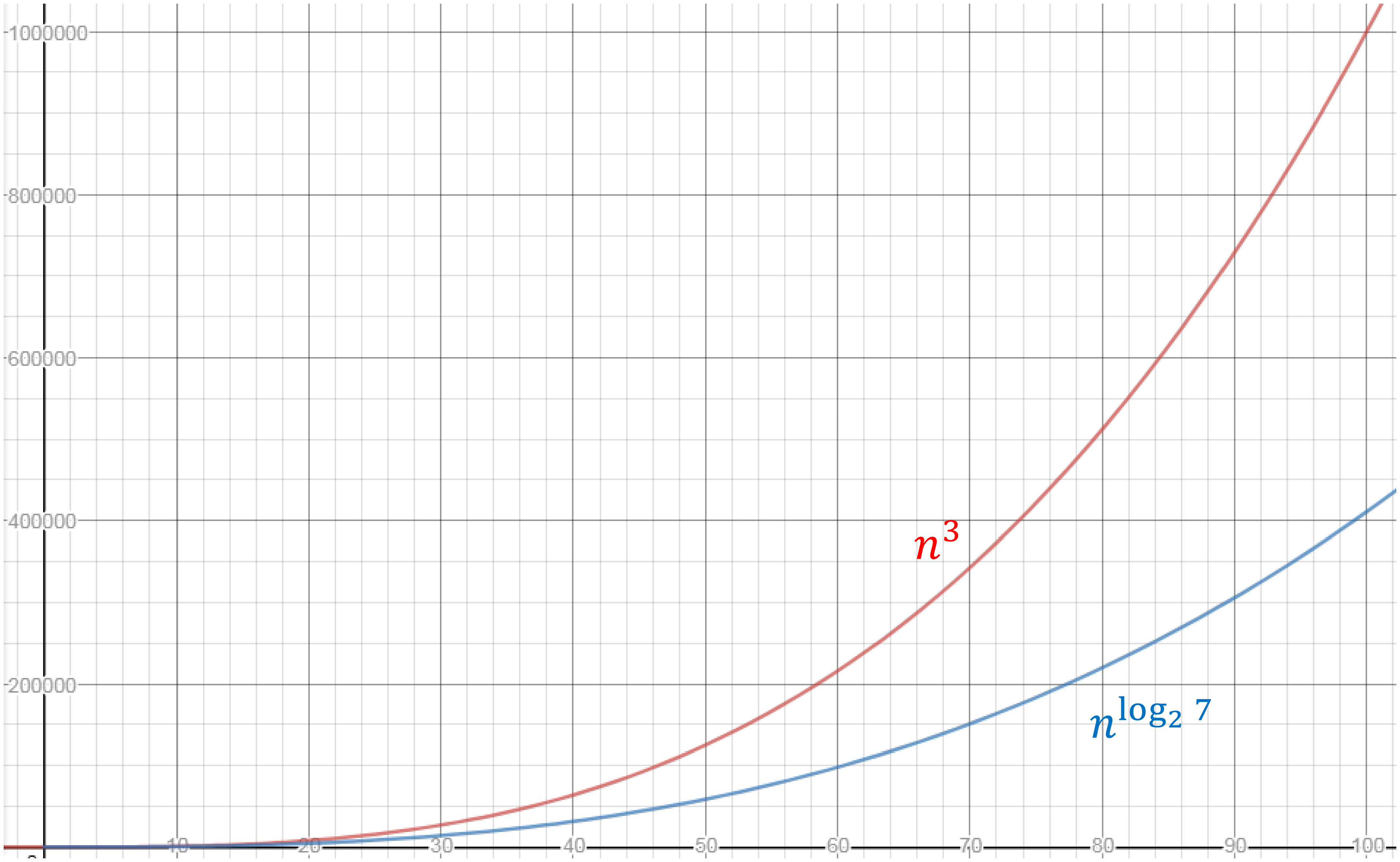
$$a = 7, b = 2, \underline{f(n) = \frac{9}{2}n^2}$$

Case 1!

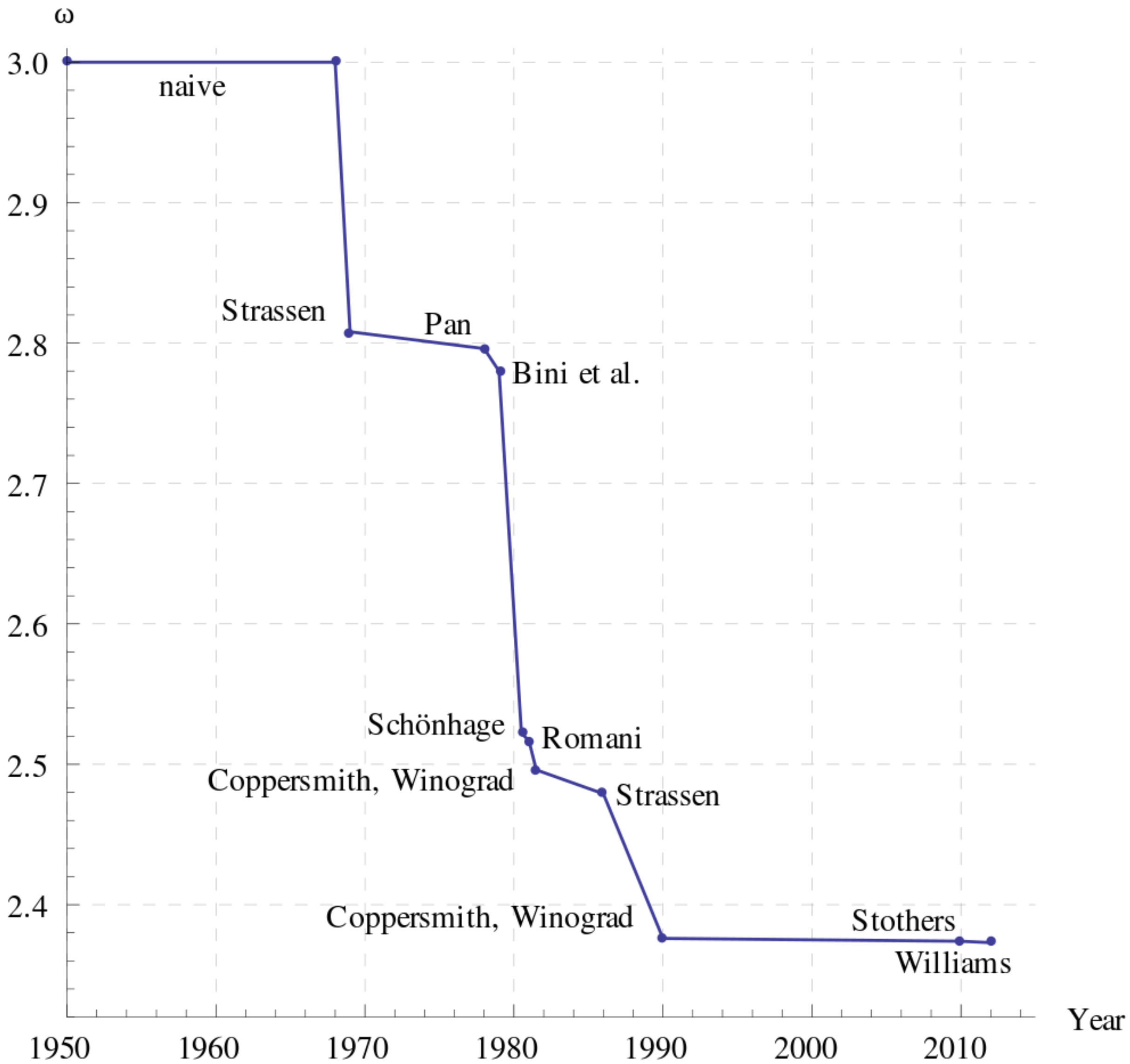
$$\underline{n^{\log_b a}} = n^{\log_2 7} \approx n^{2.\underline{807}}$$

$\varepsilon = .1$

$$T(n) = \Theta\left(n^{\log_2 7}\right) \approx \Theta(n^{2.807})$$


1000000
800000
600000
400000
200000
10
20
30
40
50
60
70
80
90
100
n^3
$n^{\log_2 7}$

Is This the Fastest?



Best possible is still unknown

Best lower bound: $\Omega(n^2)$

$n^{2,371552}$

2023

Divide and Conquer Algorithms (Thus Far)

Mergesort

Naïve Multiplication

Karatsuba Multiplication

Closest Pair of Points

Strassen's Algorithm

What they have in common:

Divide: Very easy (i.e. $O(1)$)

Combine: More complex ($\Omega(n)$)

Quicksort

Like Mergesort:

- Divide and conquer algorithm
- $O(n \log n)$ run time (on expectation)

Unlike Mergesort:

- **Divide** step is the hard part
- Typically faster than Mergesort (often is the basis of sorting algorithms in standard library implementations)

Quicksort

General idea: choose a pivot element, recursively sort two sublists around that element

Divide: select pivot element p, Partition(p)

Conquer: recursively sort left and right sublists

Combine: nothing!

Partition Procedure (Divide Step)

Input: an <u>unordered</u> list, a pivot p

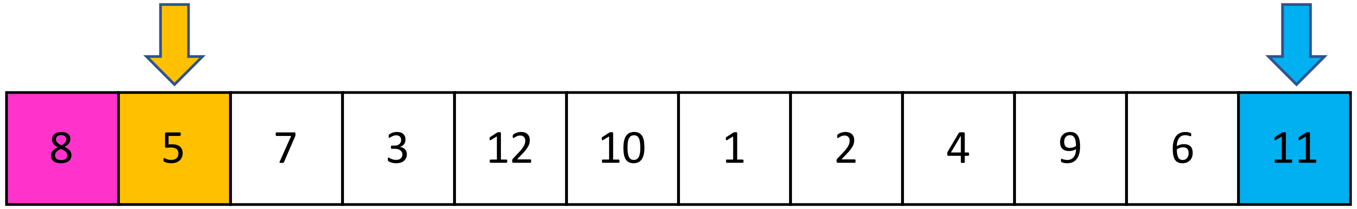
8	5	7	3	12	10	1	2	4	9	6	11

Goal: All elements $< p$ on left, all $\geq p$ on right

5	7	3	1	2	4	6	8	12	10	9	11

Partition Procedure

Initialize two pointers Begin and End

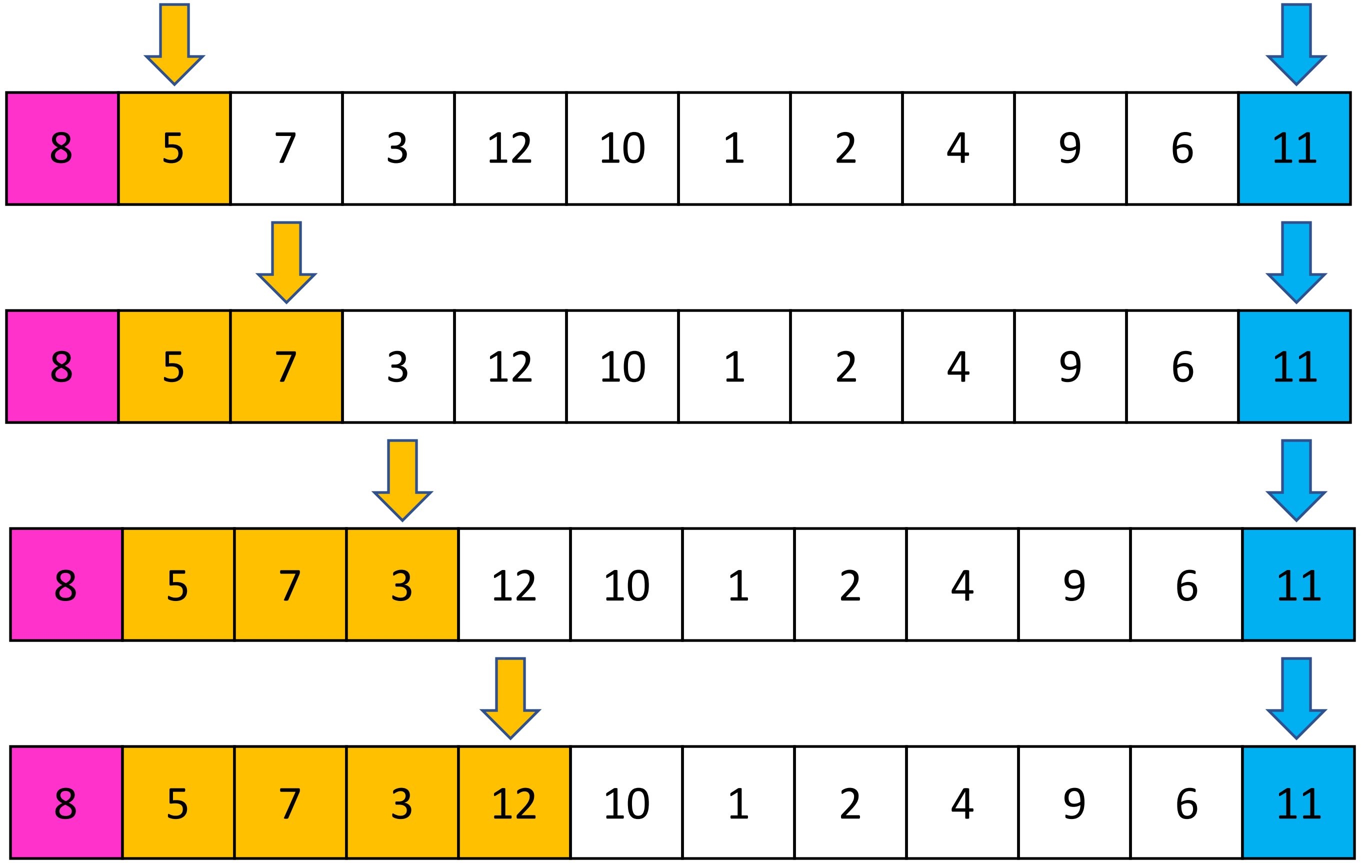


Partition Procedure

If Begin value $< \ p$, move Begin right

Else swap Begin value with End value, move End Left

Stop when Begin = End



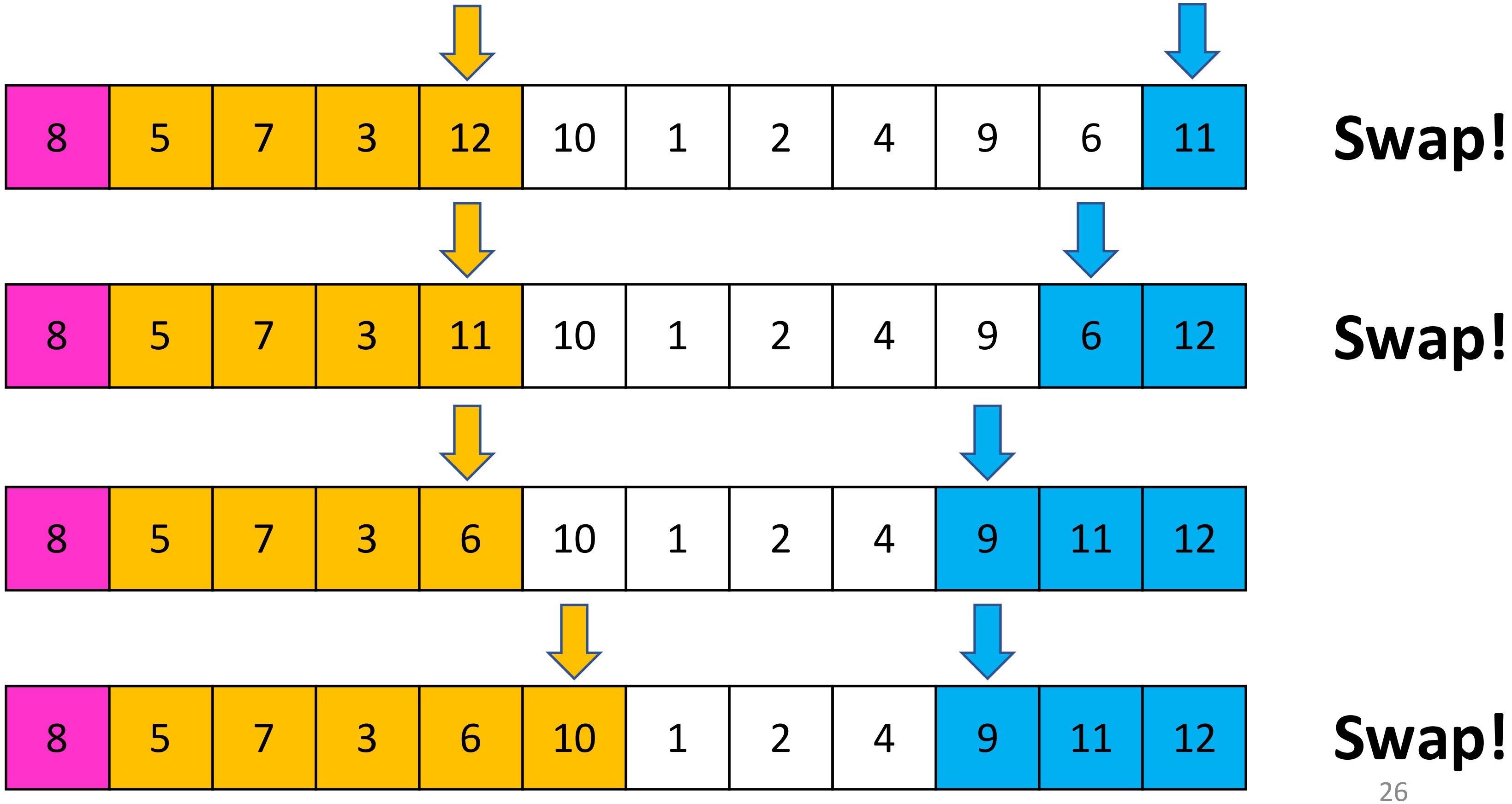
Swap!

Partition Procedure

If Begin value $<\ p$, move Begin right

Else swap Begin value with End value, move End Left

Stop when Begin = End

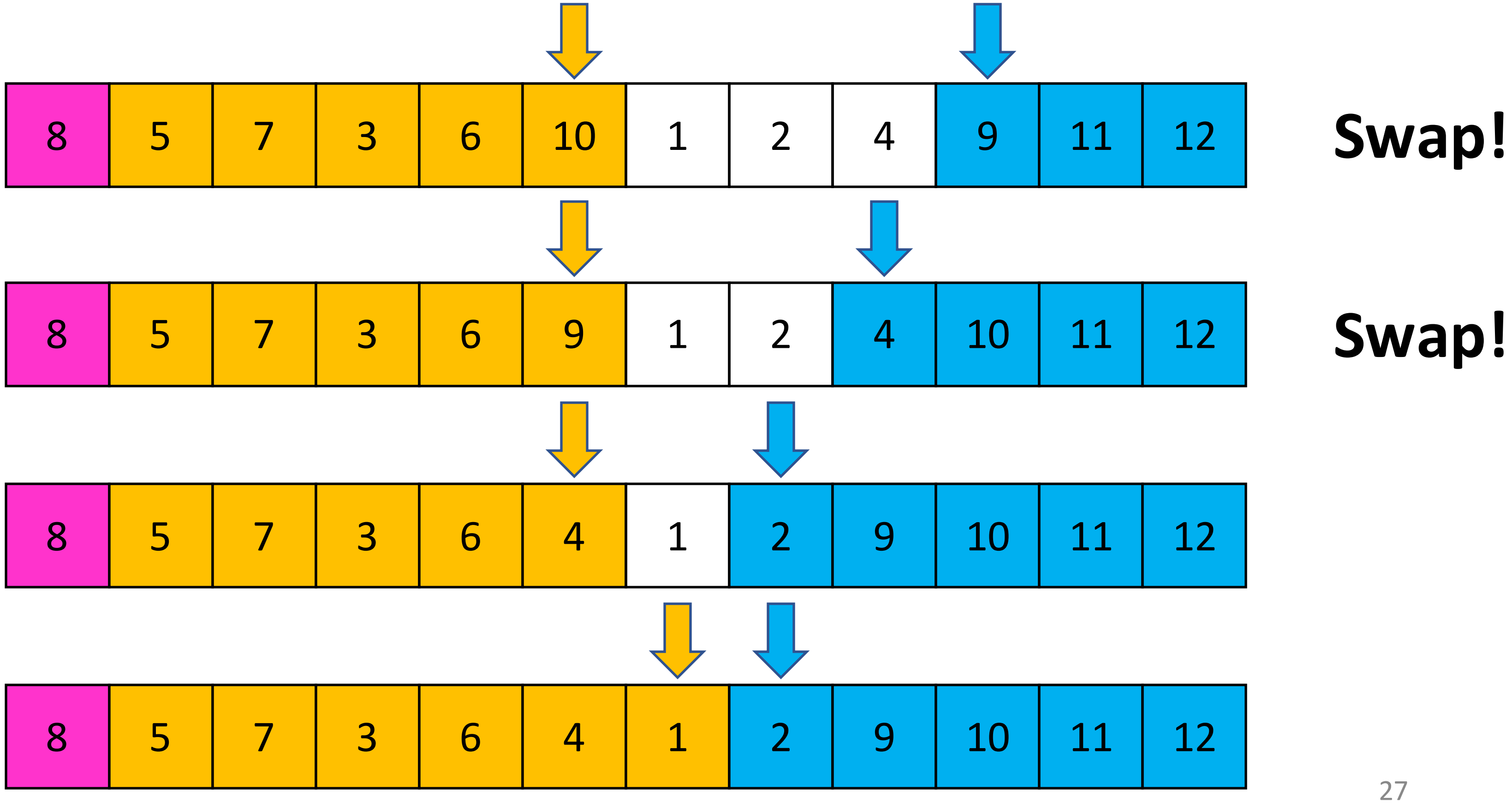


Partition Procedure

If Begin value $<\ p$, move Begin right

Else swap Begin value with End value, move End Left

Stop when Begin = End

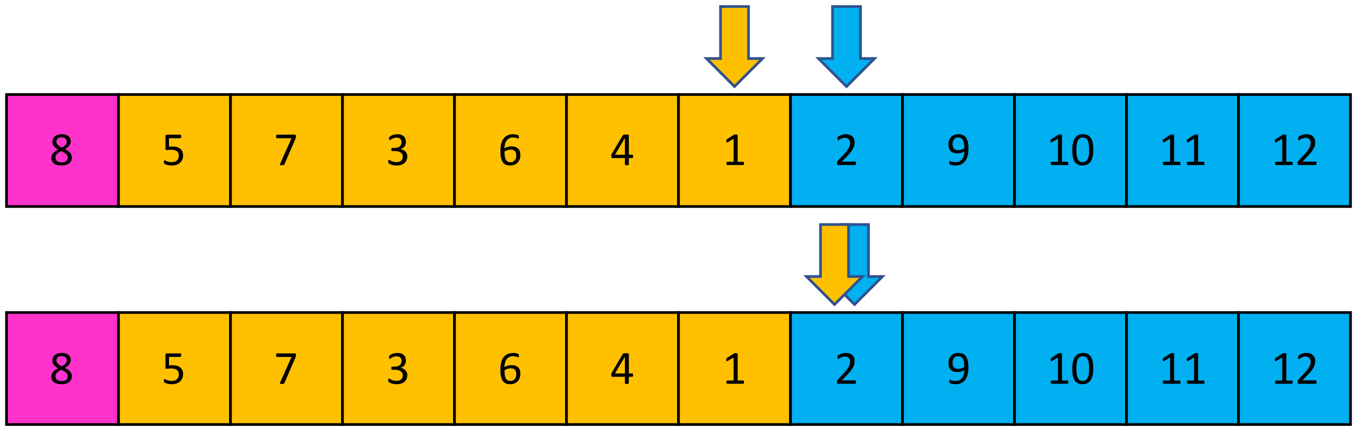


Partition Procedure

If Begin value $<\ p$, move Begin right

Else swap Begin value with End value, move End Left

Stop when Begin = End



Remaining item: where do we place the pivot?

Partition Procedure

If Begin value $<$ p, move Begin right

Else swap Begin value with End value, move End Left

Stop when Begin = End

8	5	7	3	6	4	1	2	9	10	11	12

8	5	7	3	6	4	1	2	9	10	11	12

Case 1: meet at element $< p$

Swap p with pointer position

2	5	7	3	6	4	1	8	9	10	11	12

Partition Procedure

If Begin value $<\ p$, move Begin right

Else swap Begin value with End value, move End Left

Stop when Begin = End

8	5	7	3	6	4	1	2	9	10	11	12

8	5	7	3	6	4	1	2	9	10	11	12

Case 2: meet at element $> p$

Swap p with value to the left

Partition Procedure Summary

1. Choose the pivot p to be the first element of the list
2. Initialize two pointers Begin (just after p), and End (at end of list)
3. While Begin < End:
   - If value of Begin $<$ p, advance Begin to the right
   - Otherwise, swap value of Begin value with value of End value, and advance End to the left
4. If pointers meet at element $< p$: swap p with pointer position
5. Otherwise, if pointers meet at element $> p$: swap p with value to the left

Run time? $\Theta(n)$

Conquer Step

2	5	7	3	6	4	1	8	9	10	11	12

All elements $< p$

All elements $> p$

Exactly where it belongs!

Recursively sort Left and Right sublists

Quicksort Run Time (Optimistic)

If the pivot is the median:

2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

Then we divide in half each time

$$T(n) = 2T(n/2) + n = \Theta(n \log n)$$

Quicksort Run Time (Worst-Case)

If the pivot is the extreme (min/max):

1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

Then we shorten by 1 each time

$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&= n + (n-1) + \cdots + 2 + 1 \\
&= \frac{n(n+1)}{2} = \Theta(n^2)
\end{aligned}
$$

Quicksort on a Nearly Sorted List

First element always yields unbalanced pivot

1	2	3	4	5	6	7	8	9	10	11	12

1	2	3	4	5	6	7	8	9	10	11	12

Then we shorten by 1 each time

$$T(n) = \Theta(n^2)$$

How to Choose the Pivot?

Good choice: $\Theta(n \log n)$

Bad choice: $\Theta(n^2)$

Good Pivot

What makes a good pivot?

$T(n) = 2T(\frac{n}{2}) + \underline{\Theta(n)}$

- Roughly even split between left and right
- Ideally: median

Can we find median in linear time?

- Yes! <u>Quickselect algorithm</u>

Quickselect Algorithm

Algorithm to compute the i^{th} order statistic

- i^{th} smallest element in the list
- 1st order statistic: minimum
- n^{th} order statistic: maximum
- $(n/2)^{\text{th}}$ order statistic: median

Quickselect Algorithm

Finds i^{th} order statistic

General idea: choose a pivot element, partition around the pivot, and recurse on sublist containing index i

Divide: select pivot element p, Partition(p)

Conquer:

- if $i =$ index of p, then we are done and return p
- if $i <$ index of p recurse left. Otherwise, recurse right

Combine: Nothing!

Partition Procedure (Divide Step)

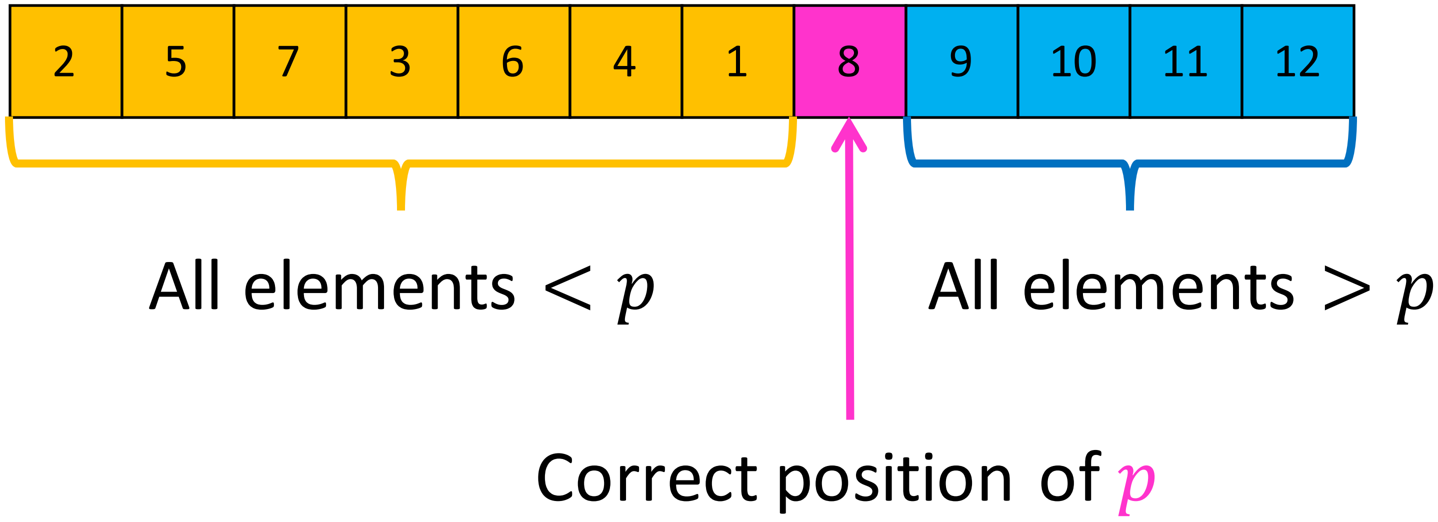
Input: an <u>unordered</u> list, a pivot p

8	5	7	3	12	10	1	2	4	9	6	11

Goal: All elements $< p$ on left, all $\geq p$ on right

5	7	3	1	2	4	6	8	12	10	9	11

Conquer Step



Recurse on sublist that contains index i

(add index of the pivot to i if recursing right)

CLRS Pseudocode for Quickselect

A – the list
p – index of first item
r – index of last item
i – find *i*th smallest item
q – pivot location
k – number on left + 1

```
RANDOMIZED-SELECT(A, p, r, i)
if p == r
    return A[p]
q = RANDOMIZED-PARTITION(A, p, r)
k = q − p + 1        // number of elements in left sub-list + 1
if i == k            // the pivot value is the answer
    return A[q]
elseif i < k
                          first   last
    return RANDOMIZED-SELECT(A, p, q − 1, i)
else return RANDOMIZED-SELECT(A, q + 1, r, i − k)
```

// note adjustment to *i* when recursing on right side

Note: In CLRS, they're using a partition that randomly chooses the pivot element.
That's why you see "Randomized" in the names here. Ignore that for the moment.

Quickselect Run Time

If the pivot is always the median:

2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

Then we divide in half each time

$a = 1$ $\quad n^{\log_2 1} = n^0 = 1$

$b = 2$ $\quad f(n) = n$

$$S(n) = S\left(\frac{n}{2}\right) + n$$

$$S(n) = O(n)$$

Quickselect Run Time

If the partition is always unbalanced:

1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

Then we shorten by 1 each time

$$S(n) = S(n-1) + n$$

$$S(n) = O(n^2)$$

How to Choose the Pivot?

Good choice: $\Theta(n)$

Bad choice: $\Theta(n^2)$

Good Pivot

What makes a good pivot?

- Roughly even split between left and right
- Ideally: median

But this is the problem that Quickselect is supposed to solve!

Déjà vu?

What's next: an algorithm for choosing a "decent" pivot (median of medians)

Good Pivot for Quickselect

What makes a good Pivot for Quickselect?

- Roughly even split between left and right
- Ideally: median

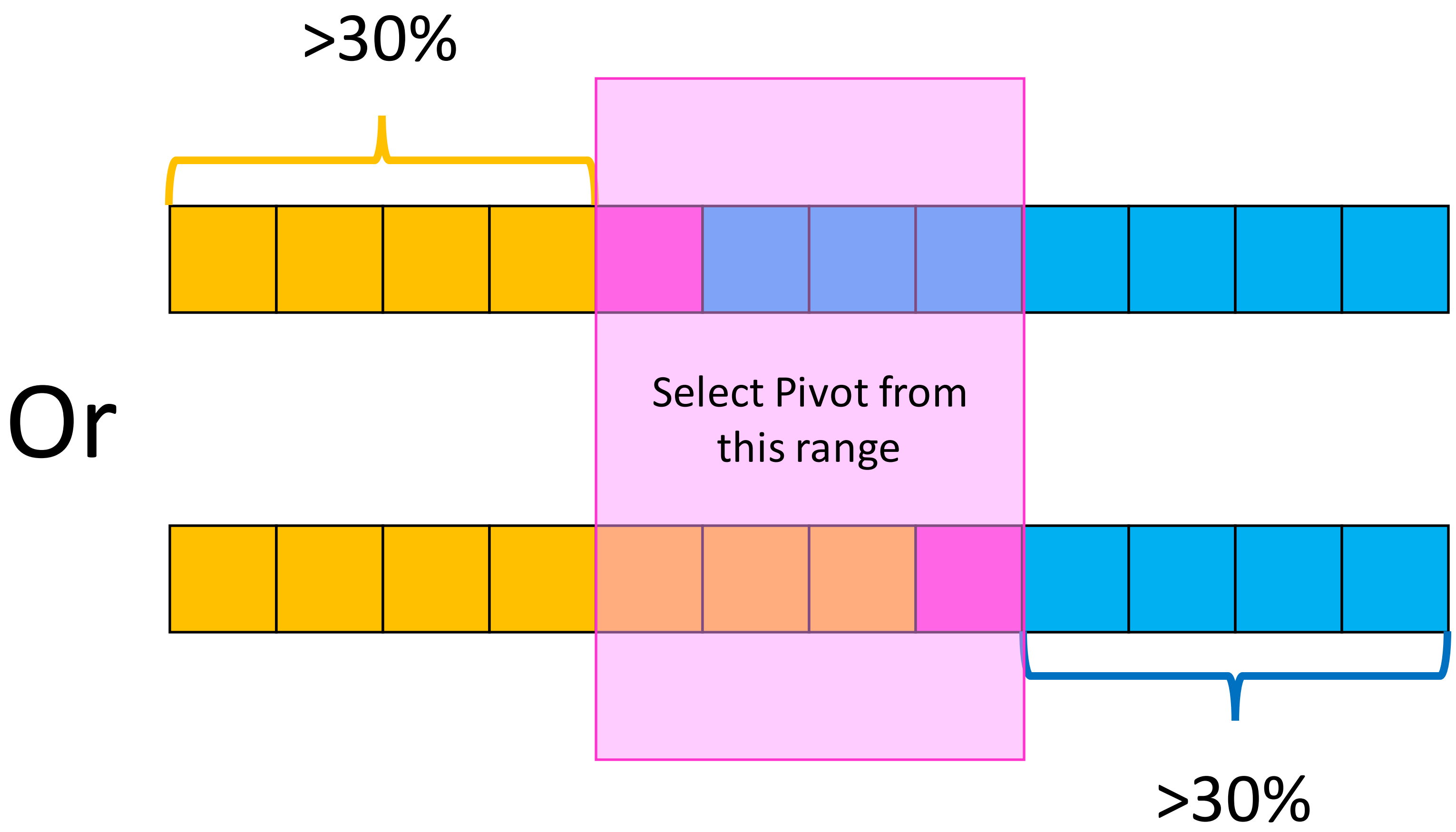
Déjà vu?

Here's what's next:

- First, **median of medians** algorithm
  - Finds something close to the median in $\Theta(n)$ time
- Second, we can prove that when its result used with Quickselect's partition, then Quickselect is guaranteed $\Theta(n)$
  - Because we now have a $\Theta(n)$ way to find the median, this guarantees Quicksort will be $\Theta(n \lg n)$
- Notes:
  - We have to do all this for every call to Partition in Quicksort
  - We could just use the value returned by median of medians for Quicksort's Partition

Good Pivot

Decent pivot: both sides of Pivot >30%



Median of Medians

Fast way to select a “good” pivot

Guarantees pivot is greater than $\approx$30% of elements and less than $\approx$30% of the elements

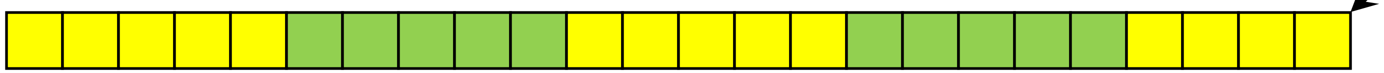
- I.e. it’s in the middle 40% (±20% of the true median)

Main idea: break list into blocks, find the median of each blocks, use the median of those medians

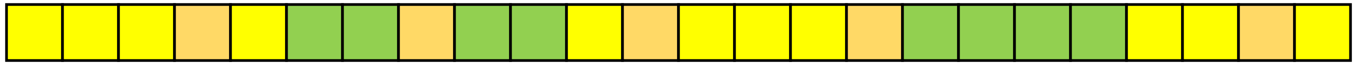
Median of Medians

1. Break list into chunks of size 5

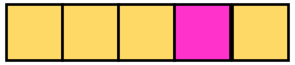
List could be long, many more than 5 chunks!



2. Find the median of each chunk
(using insertion sort: n=5, max 20 comparisons per chunk)

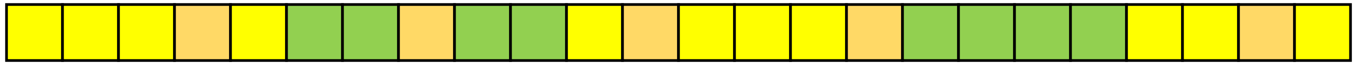


3. Return median of medians (using Quickselect, this algorithm, called recursively, on list of medians)

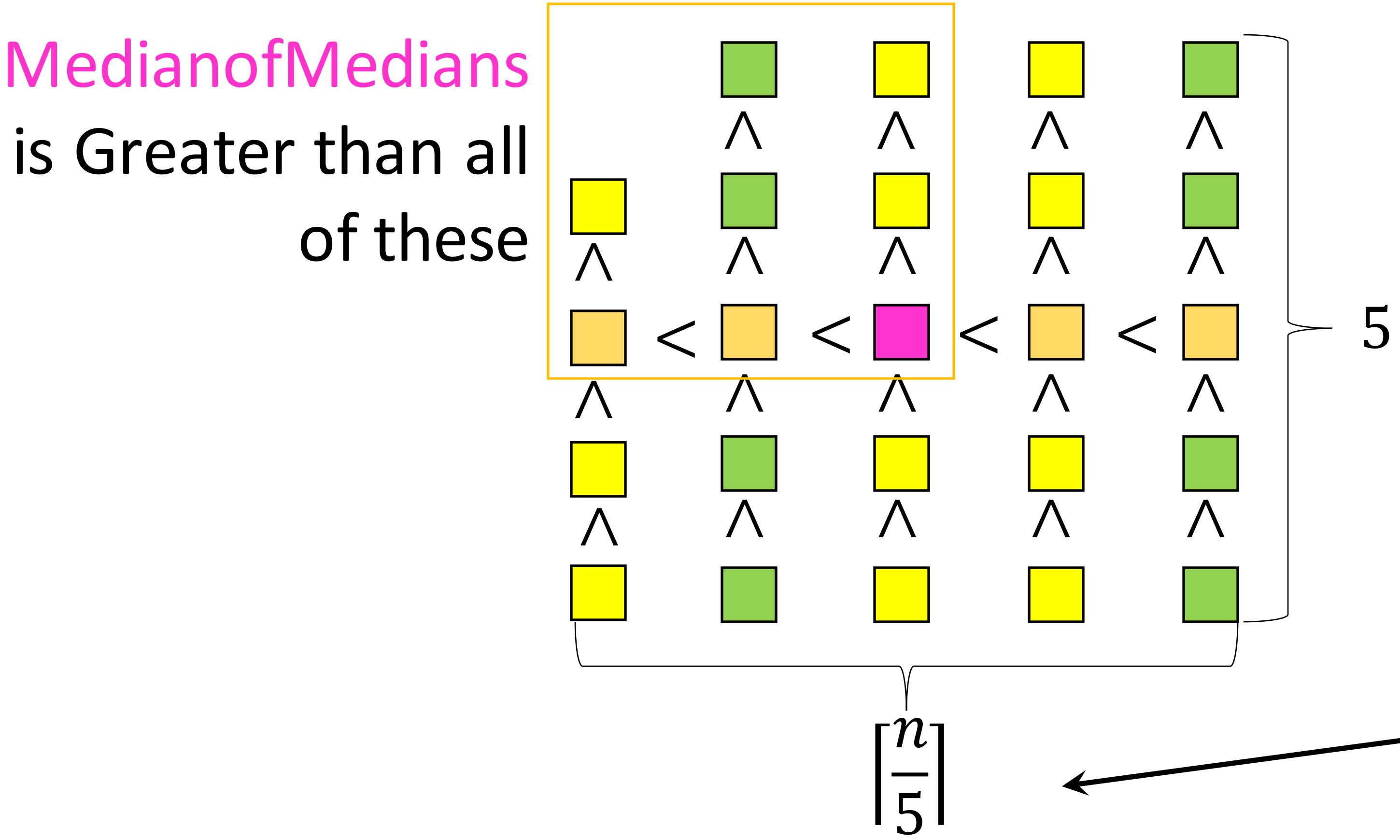


List could be long, many more than 5 medians!

Why is this good?



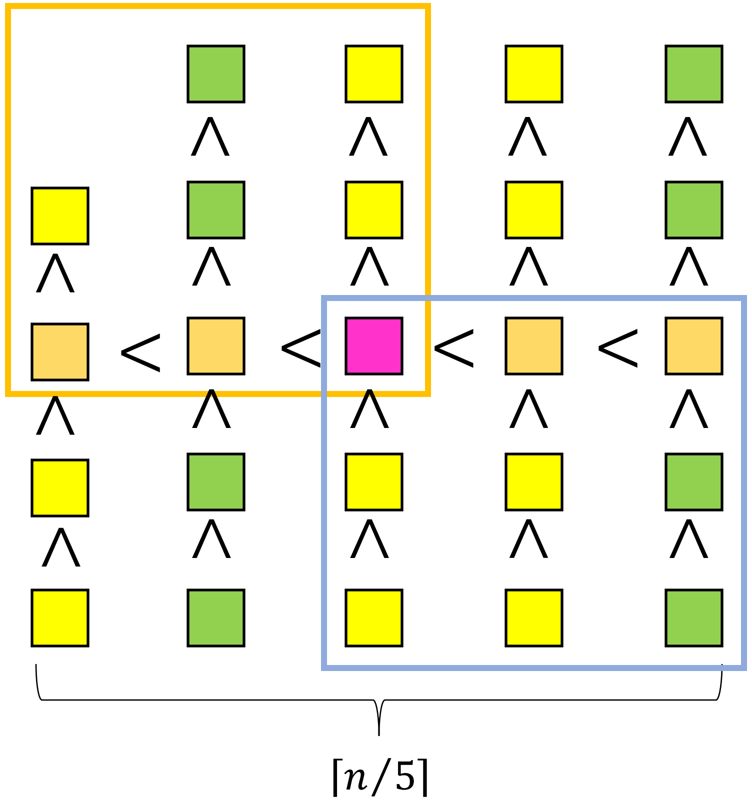
Each chunk sorted, chunks ordered by their medians



List could be long, so not a small number!

Why is this good?

MedianofMedians is larger than all of these



Elements smaller than MedianofMedians:

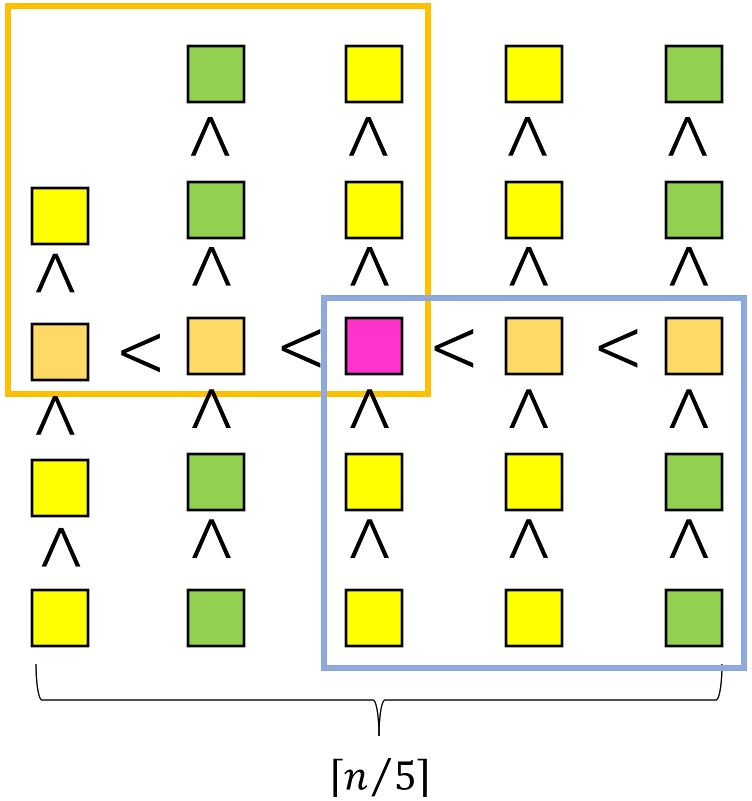
$$3\left(\left\lceil \frac{1}{2} \cdot \left\lceil \frac{n}{5} \right\rceil \right\rceil - 2\right) \geq \frac{3n}{10} - 6 \text{ elements}$$

Number of lists to the "left"

Exclude list on the endpoint, and "middle" list

Why is this good?

MedianofMedians
is larger than all
of these



Elements smaller than MedianofMedians: $3\left(\left\lceil \frac{1}{2}\cdot\left\lceil \frac{n}{5}\right\rceil\right\rceil - 2\right) \geq \frac{3n}{10} - 6$ elements

Elements greater than MedianofMedians: $3\left(\left\lceil \frac{1}{2}\cdot\left\lceil \frac{n}{5}\right\rceil\right\rceil - 2\right) \geq \frac{3n}{10} - 6$ elements

Back to: Quickselect

Divide: select an element p using Median of Medians, Partition(p)

$$M(n) + \Theta(n)$$

median of medians algorithm

partition algorithm

Quickselect

Divide: select an element p using Median of Medians, Partition(p)

$$M(n) + \Theta(n)$$

Conquer: if $i =$ index of p, done, if $i <$ index of p recurse left. Else recurse right (with index $i - p$)

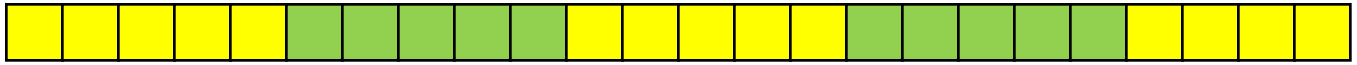
$$\leq S\left(\frac{7n}{10}\right)$$

Combine: Nothing!

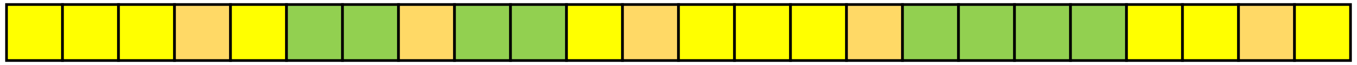
$$S(n) \leq S\left(\frac{7n}{10}\right) + M(n) + \Theta(n)$$

Median of Medians

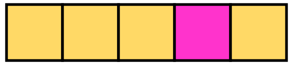
1. Break list into blocks of size 5 $\Theta(n)$



2. Find the median of each chunk $\Theta(n)$



3. Return median of medians (using Quickselect) $S\left(\frac{n}{5}\right)$



$$M(n) = S\left(\frac{n}{5}\right) + \Theta(n)$$

Quickselect

$$S(n) \le S\left(\frac{7n}{10}\right) + M(n) + \Theta(n)$$

$$M(n) = S\left(\frac{n}{5}\right) + \Theta(n)$$

$$= S\left(\frac{7n}{10}\right) + S\left(\frac{n}{5}\right) + \Theta(n)$$

$$= S\left(\frac{7n}{10}\right) + S\left(\frac{2n}{10}\right) + \Theta(n)$$

$$\le S\left(\frac{9n}{10}\right) + \Theta(n)$$ Because $S(n) = \Omega(n)$

CLRS gives a more rigorous proof!
See p. 203 for more details

Master theorem Case 3!

$$S(n) = O(n)$$

$$S(n) = \Theta(n)$$

Phew! Back to Quicksort

Divide: Select a pivot element, and partition about the pivot

2	5	1	3	6	4	7	8	10	9	11	12

Using Quickselect, always pivot about the median

2	1	3	5	6	4	7	8	9	10	11	12

Conquer: Recursively sort left and right sublists

If pivot is the median, list is split in half each iteration

Phew! Back to Quicksort

Divide: Select a pivot element, and partition about the pivot

2	5	1	3	6	4	7	8	10	9	11	12

Using Quickselect, always pivot about the median

2	1	3	5	6	4	7	8	9	10	11	12

$$T(n) = 2T(n/2) + \Theta(n)$$

$$T(n) = \Theta(n \log n)$$

A Worthwhile Choice?

Using Quickselect to pick median guarantees $\Theta(n \log n)$ worst-case run-time

Approach has very large constants

- If you really want $\Theta(n \log n)$, better off using MergeSort

More efficient approach: Random pivot

- Very small constant (very fast algorithm)
- Expected to run in $\Theta(n \log n)$ time
  - Why? Unbalanced partitions are very unlikely

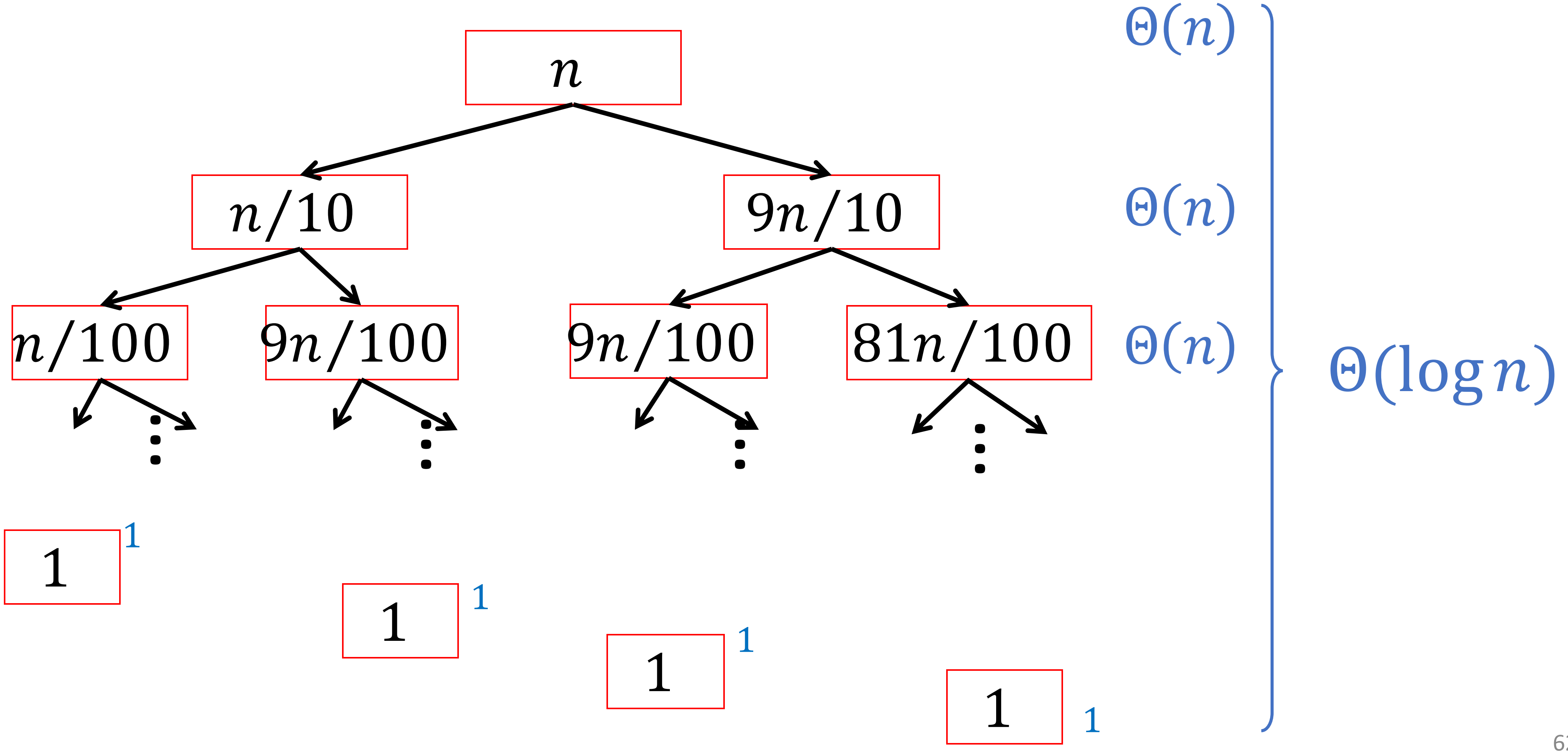
Quicksort Running Time

If the pivot is always $(n/10)^{\text{th}}$ order statistic:

$$T(n) = T(n/10) + T(9n/10) + \Theta(n)$$

Quicksort Running Time

$$T(n) = T(n/10) + T(9n/10) + \Theta(n)$$



Quicksort Running Time

If the pivot is always $(n/10)^{\text{th}}$ order statistic:

$$T(n) = T(n/10) + T(9n/10) + \Theta(n)$$
$$= \Theta(n \log n)$$

This is true if the pivot is any $(n/k)^{\text{th}}$ order statistic for any constant $k > 1$ (as long as the size of the smaller list is a <u>constant fraction</u> of the full list, we get $\Theta(n \log n)$ running time)

Quicksort Running Time

If the pivot is always d^{th} order statistic:

1	5	2	3	6	4	7	8	10	9	11	12

Then we shorten by d each time

$$T(n) = T(n-d) + n$$
$$= \Theta(n^2)$$

What's the probability of this occurring (for a random pivot)?

Probability of Always Choosing d^{th} Order Statistic

We must consistently select pivot from within the first d terms

Probability first pivot is among d smallest: $\frac{d}{n}$

Probability second pivot is among d smallest: $\frac{d}{n-d}$

Probability all pivots are among d smallest:

Very small probability!

$$\frac{d}{n} \times \frac{d}{n-d} \times \frac{d}{n-2d} \times \cdots \times \frac{d}{2d} \times 1 = \left(\frac{n}{d} \times \left(\frac{n}{d} - 1\right) \times \cdots \times 1\right)^{-1} = \frac{1}{\left(\frac{n}{d}\right)!}$$

Formal Argument for $n \log n$ Average

We will focus on counting the number of <u>comparisons</u>

For simplicity: suppose all elements are <u>distinct</u>

Quicksort only compares against a pivot

- Element i only compared to element j if one of them was the pivot

Formal Argument for $n \log n$ Average

What is the probability of comparing two given elements?

1	2	3	4	5	6	7	8	9	10	11	12

Consider the sorted version of the list

Observation: Adjacent elements must be compared

- **Why?** Otherwise I would not know their order
- **Every** sorting algorithm **must** compare adjacent elements

In quicksort: adjacent elements <u>always</u> end up in same sublist, unless one is the pivot

Formal Argument for $n \log n$ Average

What is the probability of comparing two given elements?

1	2	3	4	5	6	7	8	9	10	11	12

Consider the sorted version of the list

$$\Pr[\text{we compare 1 and 12}] = \frac{2}{12}$$

Assuming pivot is chosen uniformly at random

Elements only compared if 1 or 12 was chosen as the first pivot since otherwise they are in <u>different</u> sublists

Formal Argument for $n \log n$ Average

What is the probability of comparing two given elements?

1	2	3	4	5	6	7	8	9	10	11	12
				i				j			

Case 1: Pivot less than i

Then sublist $[i, i+1, \dots, j]$ will be in right sublist and will be processed in future invocation of Quicksort

$$\Pr[\text{we compare } i \text{ and } j] = \Pr[\text{we compare } i \text{ and } j \text{ in Quicksort}([p+1, \dots, n])]$$

Formal Argument for $n \log n$ Average

What is the probability of comparing two given elements?

1	2	3	4	5	6	7	8	9	10	11	12
				i				j			

Case 1: Pivot less than i

Then sublist $[i, i+1, \dots, j]$ will be

processed in future invocation of

$[p+1, \dots, n]$ denotes the right sublist (in some order) that we are recursively sorting

$$\Pr[\text{we compare } i \text{ and } j] = \Pr[\text{we compare } i \text{ and } j \text{ in Quicksort}([p+1, \dots, n])]$$

Formal Argument for $n \log n$ Average

What is the probability of comparing two given elements?

1	2	3	4	5	6	7	8	9	10	11	12
				i				j			

Case 2: Pivot greater than j

Then sublist $[i, i+1, \dots, j]$ will be in left sublist and will be processed in future invocation of Quicksort

$$\Pr[\text{we compare } i \text{ and } j] = \Pr[\text{we compare } i \text{ and } j \text{ in Quicksort}([1, \dots, p])]$$

Formal Argument for $n \log n$ Average

What is the probability of comparing two given elements?

1	2	3	4	5	6	7	8	9	10	11	12
				i				j			

Case 3.1: Pivot contained in $[i+1, \dots, j-1]$

Then i and j are in different sublists and will <u>never</u> be compared

$$\Pr[\text{we compare } i \text{ and } j] = 0$$

Formal Argument for $n \log n$ Average

What is the probability of comparing two given elements?

1	2	3	4	5	6	7	8	9	10	11	12
				i				j			

Case 3.2: Pivot is either i or j

Then we will always compare i and j

$$\Pr[\text{we compare } i \text{ and } j] = 1$$

Formal Argument for $n \log n$ Average

What is the probability of comparing two given elements?

1	2	3	4	5	6	7	8	9	10	11	12
				i				j			

Case 1: Pivot less than i

$$\Pr[\text{we compare } i \text{ and } j] = \Pr[\text{we compare } i \text{ and } j \text{ in Quicksort}([p+1, \ldots, n])]$$

Case 2: Pivot greater than j

$$\Pr[\text{we compare } i \text{ and } j] = \Pr[\text{we compare } i \text{ and } j \text{ in Quicksort}([1, \ldots, p])]$$

Case 3: Pivot in $[i, i+1, \ldots, j]$

$$\Pr[\text{we compare } i \text{ and } j] = \Pr[i \text{ or } j \text{ is selected as pivot}] = \frac{2}{j-i+1}$$

Formal Argument for $n \log n$ Average

Probability of comparing element i with element j:

$$\Pr[\text{we compare } i \text{ and } j] = \frac{2}{j - i + 1}$$

Formal Argument for $n \log n$ Average

Probability of comparing element i with element j:

$$\Pr[\text{we compare } i \text{ and } j] = \frac{2}{j - i + 1}$$

<u>Expected</u> number of comparisons:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{1}{k} < 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k}$$

Substitution:
$k = j - i$

$$\frac{1}{k+1} < \frac{1}{k}$$

Formal Argument for $n \log n$ Average

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{1}{k} < 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k}$$

Substitution:
$k = j - i$

$$\frac{1}{k+1} < \frac{1}{k}$$

Useful fact: $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n)$

Intuition (not proof!):
$$\sum_{k=1}^{n} \frac{1}{k} \approx \int_1^n \frac{1}{x} dx = \ln n$$

Formal Argument for $n \log n$ Average

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{1}{k} < 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k}$$

$$= 2 \sum_{i=1}^{n-1} \Theta(\log n) = \Theta(n \log n)$$

Useful fact: $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n)$