CS 3100
Data Structures and Algorithms 2
Lecture 18: Seam Carving

Co-instructors: Robbie Hott and Tom Horton

Fall 2023

Readings in CLRS 4th edition:

- Chapter 14

Announcements

- Upcoming dates
  - PA3 (Clustering) due October 29, 2023 at 11:59pm
  - PS4 (Dynamic Programming), due November 2, 2023 at 11:59pm
  - PA4 (Seam Carving) due November 12, 2023 at 11:59pm
  - Quizzes 3-4 (Greedy, Dynamic Programming) on November 9, 2023 in class
- Updated Late Policy!
  - You must submit an extension request **before** the deadline
  - Explain why need you need the extension (up to 48 hours past the deadline)
  - Acknowledge that you’re getting an extension
    - The late deadline is not the real deadline ☺
  - You may then take the additional 48 hours as needed
- Course email (comes to both professors and head TAs):

cs3100@cshelpdesk.atlassian.net

Dynamic Programming

- Requires Optimal Substructure
  - Solution to larger problem contains the (optimal) solutions to smaller ones
- Idea:
  1. Identify the recursive structure of the problem
     - What is the “last thing” done?
  2. Save the solution to each subproblem in memory
  3. Select a good order for solving subproblems
     - “Top Down”: Solve each recursively
     - “Bottom Up”: Iteratively solve smallest to largest

Log Cutting

Given a log of length n
A list (of length n) of prices P ($P[i]$ is the price of a cut of size i)
Find the best way to cut the log

Price:	1	5	8	9	10	17	17	20	24	30
Length:	1	2	3	4	5	6	7	8	9	10

Select a list of lengths $\ell_1, \ldots, \ell_k$ such that:

$\sum \ell_i = n$

to maximize $\sum P[\ell_i]$

Brute Force: $O(2^n)$

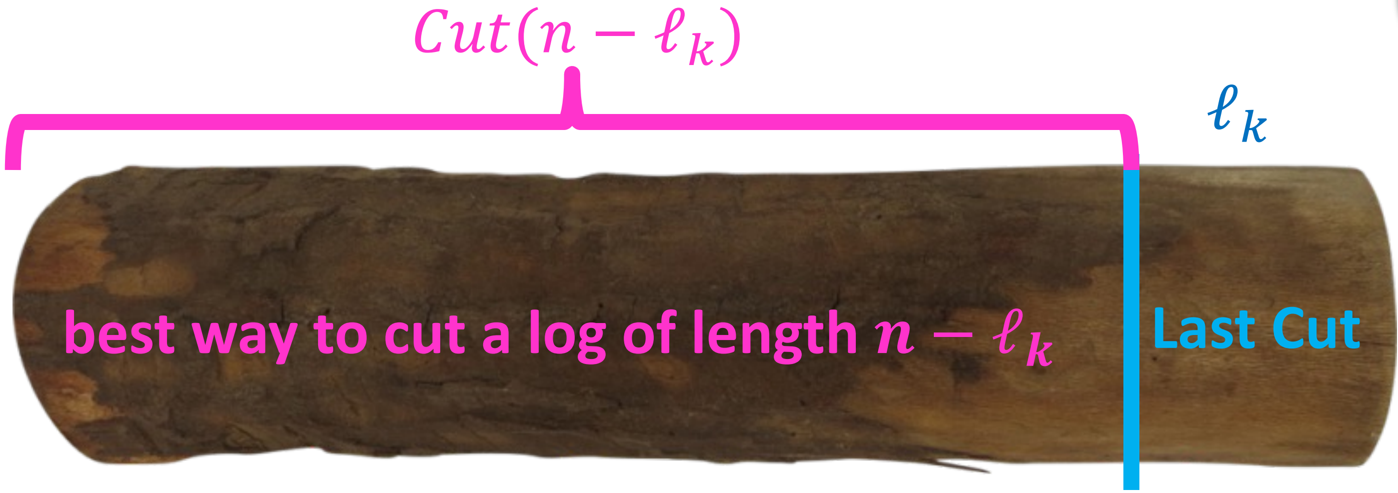
1. Identify Recursive Structure

$P[i]$ = value of a cut of length i

$Cut(n)$ = value of best way to cut a log of length n

$$Cut(n) = \max \begin{cases} Cut(n-1) + P[1] \\ Cut(n-2) + P[2] \\ \dots \\ Cut(0) + P[n] \end{cases}$$

2. Save sub-solutions to memory!



3. Select a Good Order for Solving Subproblems

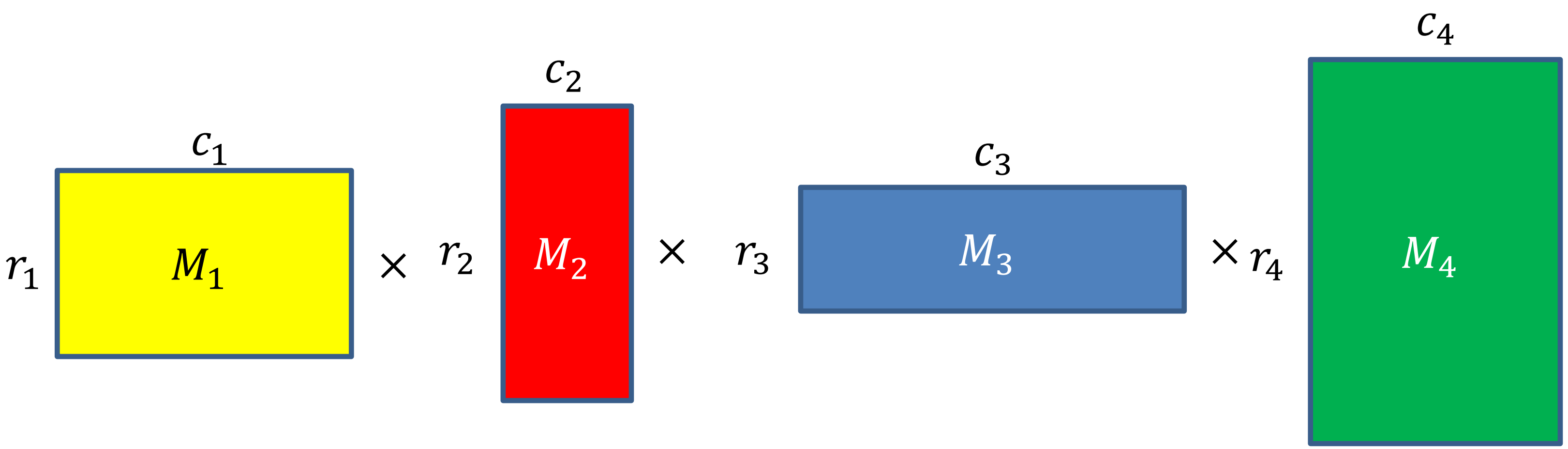
Solve Smallest subproblem first

$$Cut(4) = \max \begin{cases} Cut(3) + P[1] \\ Cut(2) + P[2] \\ Cut(1) + P[3] \\ Cut(0) + P[4] \end{cases}$$

$Cut(i)$:	0				🟨						
Length:	0	1	2	3	4	5	6	7	8	9	10

Matrix Chaining

- Given a sequence of Matrices $(M_1, \ldots, M_n)$, what is the most efficient way to multiply them?



1. Identify the Recursive Structure of the Problem

- In general:

$Best(i,j) =$ cheapest way to multiply together M_i through M_j

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$$Best(1,n) = \min \left\{ \begin{array}{l} Best(2,n) + r_1 r_2 c_n \\ Best(1,2) + Best(3,n) + r_1 r_3 c_n \\ Best(1,3) + Best(4,n) + r_1 r_4 c_n \\ Best(1,4) + Best(5,n) + r_1 r_5 c_n \\ \quad \ldots \\ Best(1,n-1) + r_1 r_n c_n \end{array} \right.$$

2. Save Subsolutions in Memory

- In general:

$Best(i,j)$ = cheapest way to multiply together M_i through M_j

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

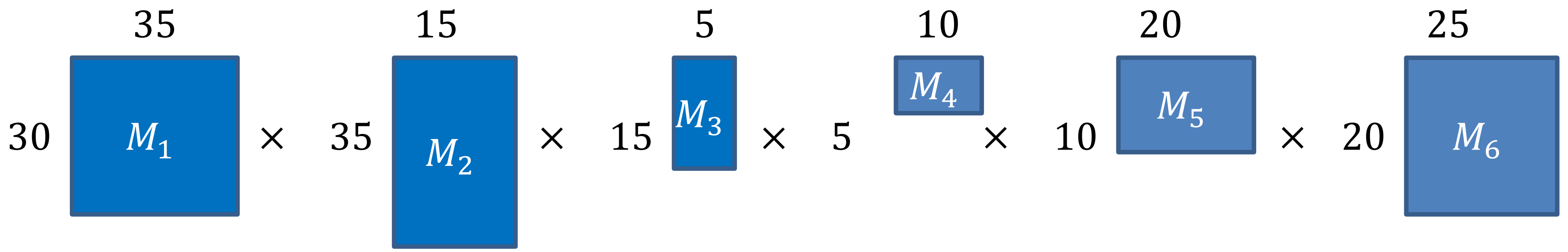
$$Best(i,i) = 0$$

Read from M[n] if present

Save to M[n]

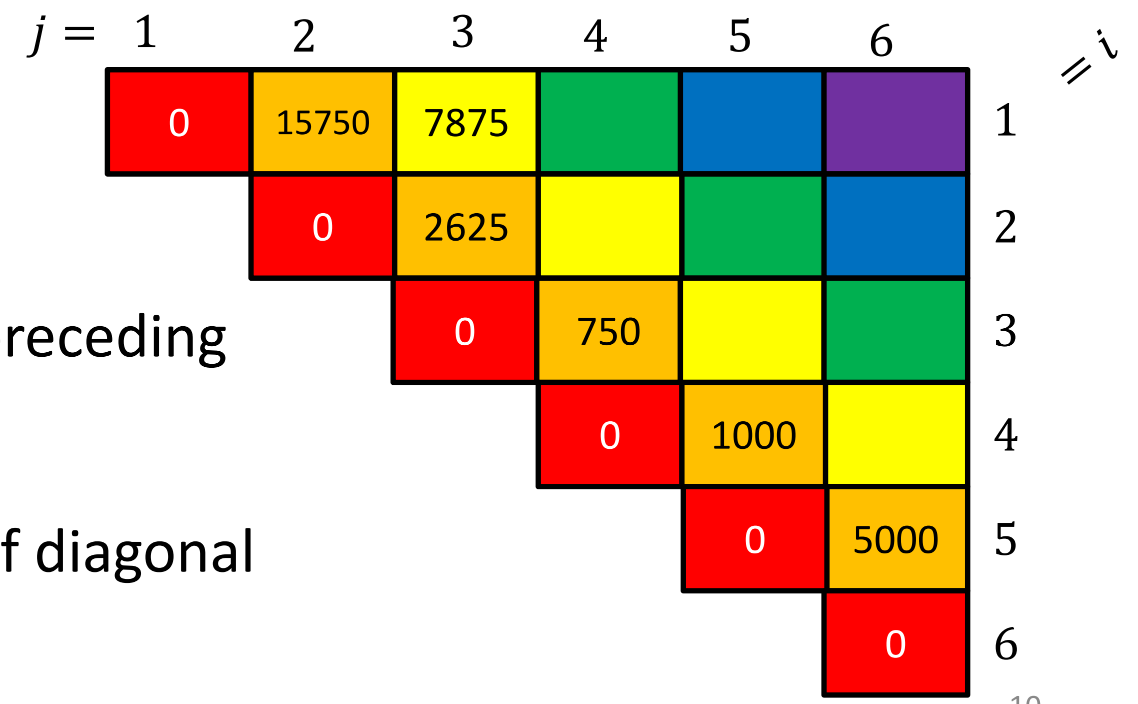
$$Best(1,n) = \min \begin{cases} Best(2,n) + r_1 r_2 c_n \\ Best(1,2) + Best(3,n) + r_1 r_3 c_n \\ Best(1,3) + Best(4,n) + r_1 r_4 c_n \\ Best(1,4) + Best(5,n) + r_1 r_5 c_n \\ \dots \\ Best(1,n-1) + r_1 r_n c_n \end{cases}$$

3. Select a good order for solving subproblems



$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$



$j =$	1	2	3	4	5	6	$= i$
	0	15750	7875	🟩	🟦	🟪	1
		0	2625	🟨	🟩	🟦	2
			0	750	🟨	🟩	3
				0	1000	🟨	4
					0	5000	5
						0	6

To find $Best(i,j)$: Need all preceding terms of row i and column j

Conclusion: solve in order of diagonal

Seinfeld break

Movie Time!

In Season 9 Episode 7 “The Slicer” of the hit 90s TV show *Seinfeld*, George discovers that, years prior, he had a heated argument with his new boss, Mr. Kruger. This argument ended in George throwing Mr. Kruger’s boombox into the ocean. How did George make this discovery?

https://www.youtube.com/watch?v=pSB3HdmLcY4

Seam Carving

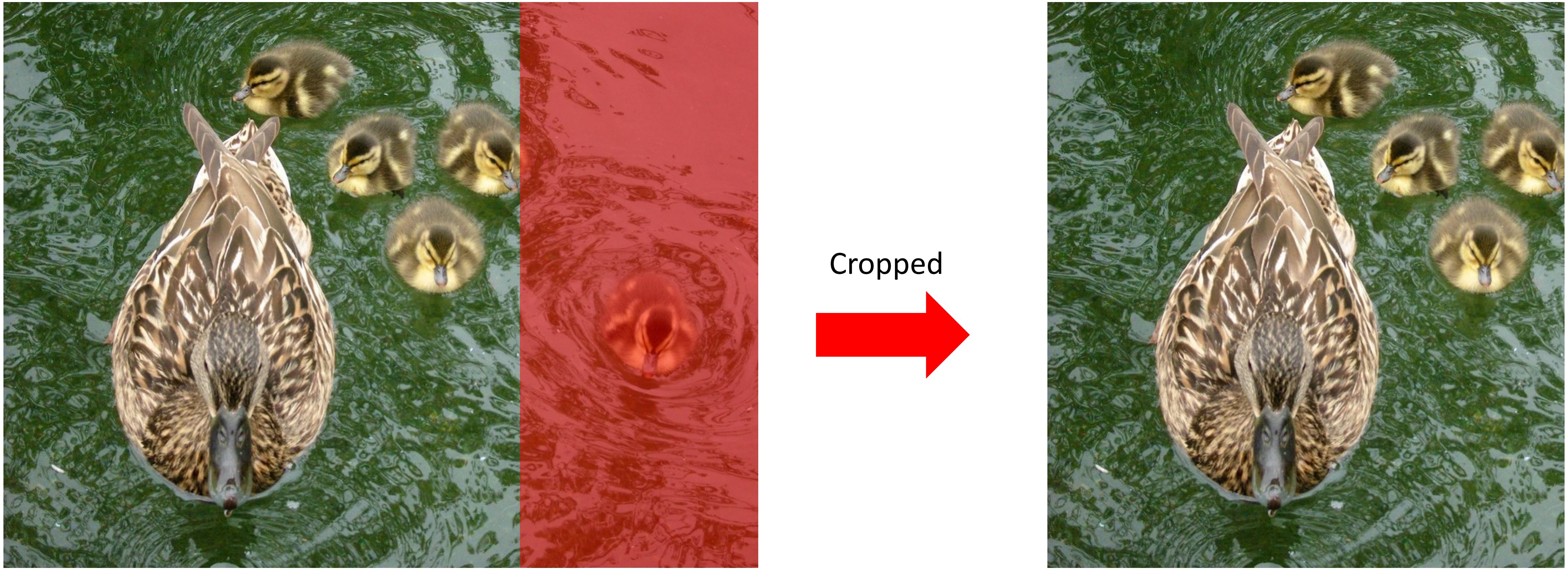
- Method for image resizing that doesn’t scale/crop the image

Seam Carving

- Method for image resizing that doesn’t scale/crop the image

Cropping

- Removes a “block” of pixels



Cropped

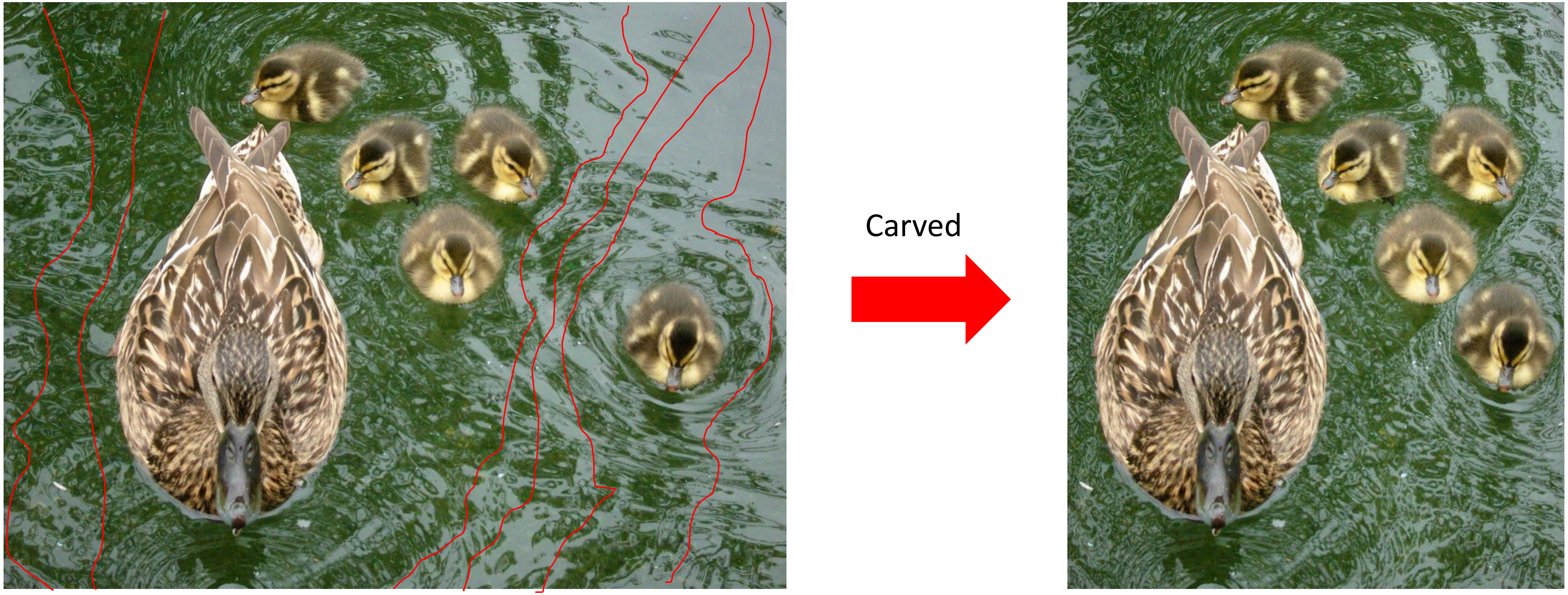
Scaling

- Removes “stripes” of pixels

Scaled

Seam Carving

- Removes “least energy seam” of pixels
- https://trekhleb.dev/js-image-carver/



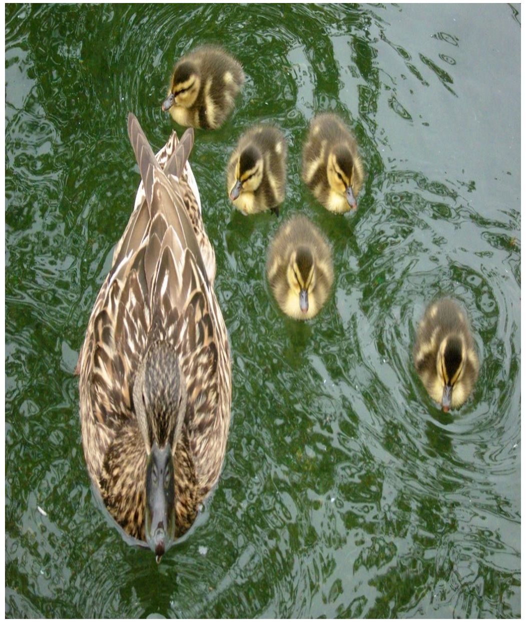
Carved

Seam Carving

- Method for image resizing that doesn’t scale/crop the image

Cropped

Scaled



Carved

Seattle Skyline

Energy of a Seam

- Sum of the energies of each pixel

$$e(p) = \text{energy of pixel } p$$

- Many choices for pixel energy
  - E.g.: change of gradient (how much the color of this pixel differs from its neighbors)
  - Particular choice doesn’t matter, we use it as a “black box”

- Goal: find least-energy seam to remove

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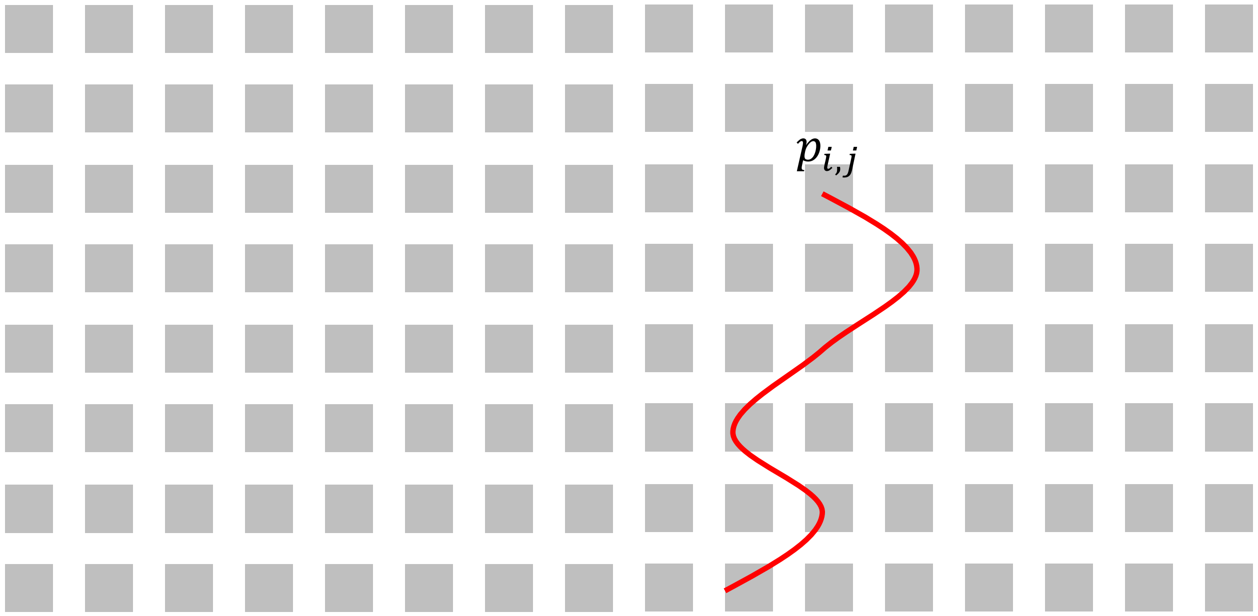
Identify Recursive Structure

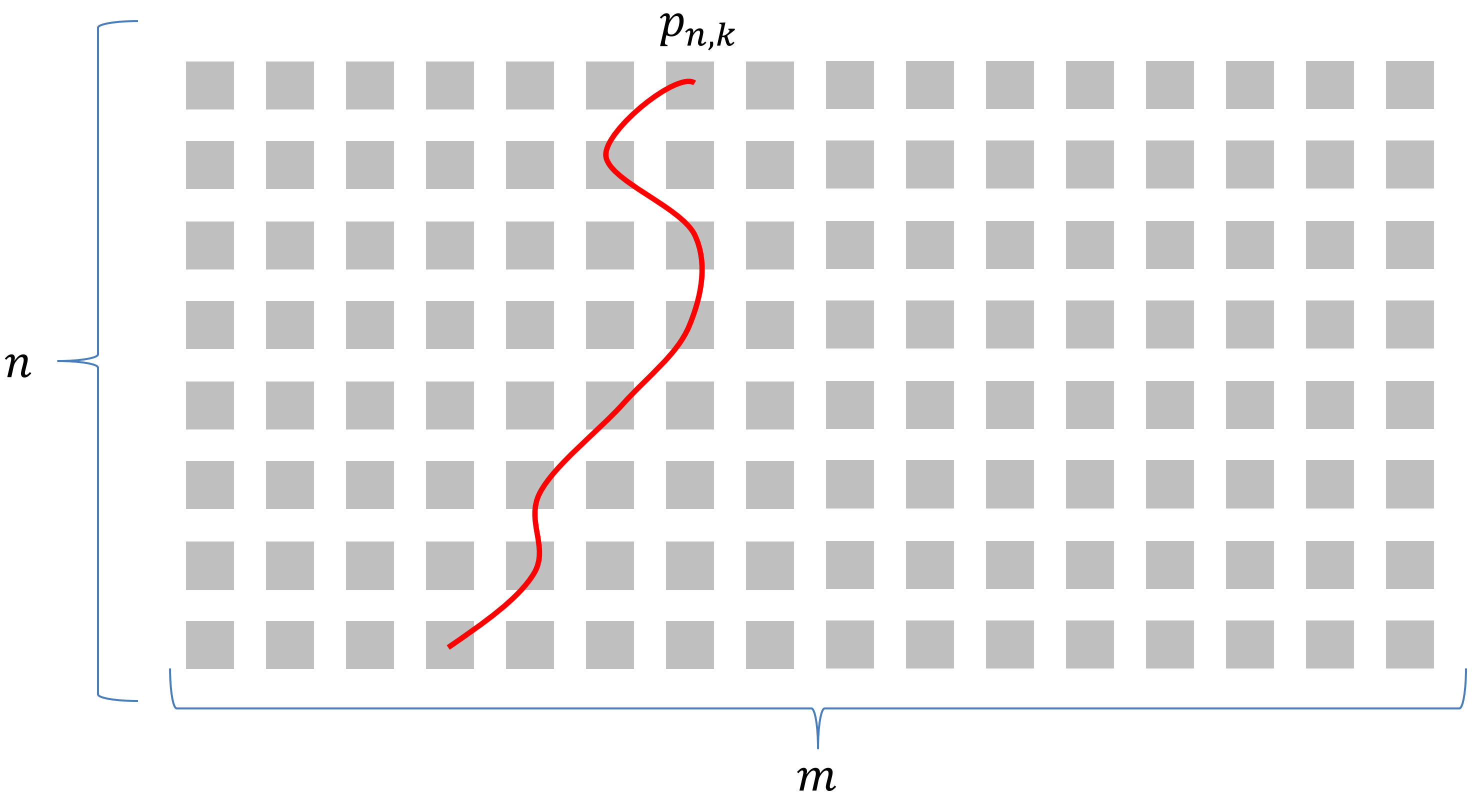
Let $S(i, j)$ = least energy seam from the bottom of the image up to pixel $p_{i,j}$



Finding the Least Energy Seam

Want to delete the least energy seam going from bottom to top, so delete:

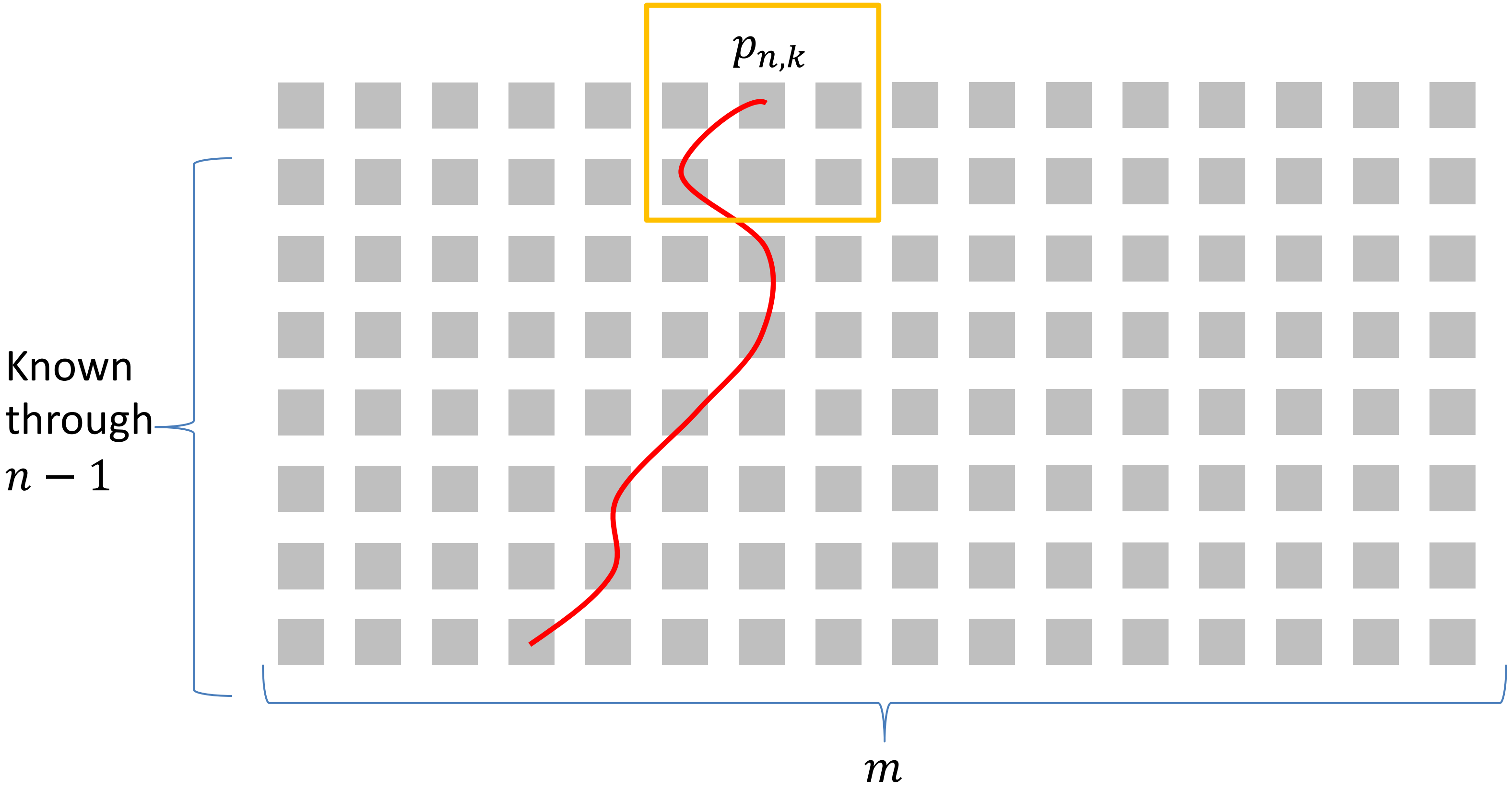
$$\min_{k=1}^{m}(S(n,k))$$



Computing $S(n, k)$

Assume we know the least energy seams for all of row $n-1$

(i.e. we know $S(n-1, \ell)$ for all ℓ)



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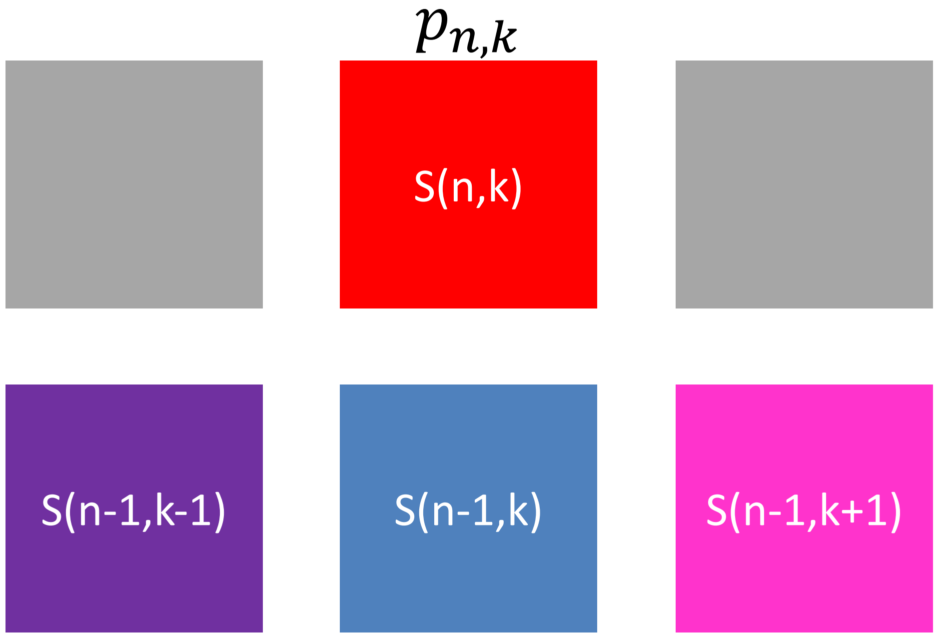


$p_{n,k}$

S(n,k)

S(n-1,k-1)

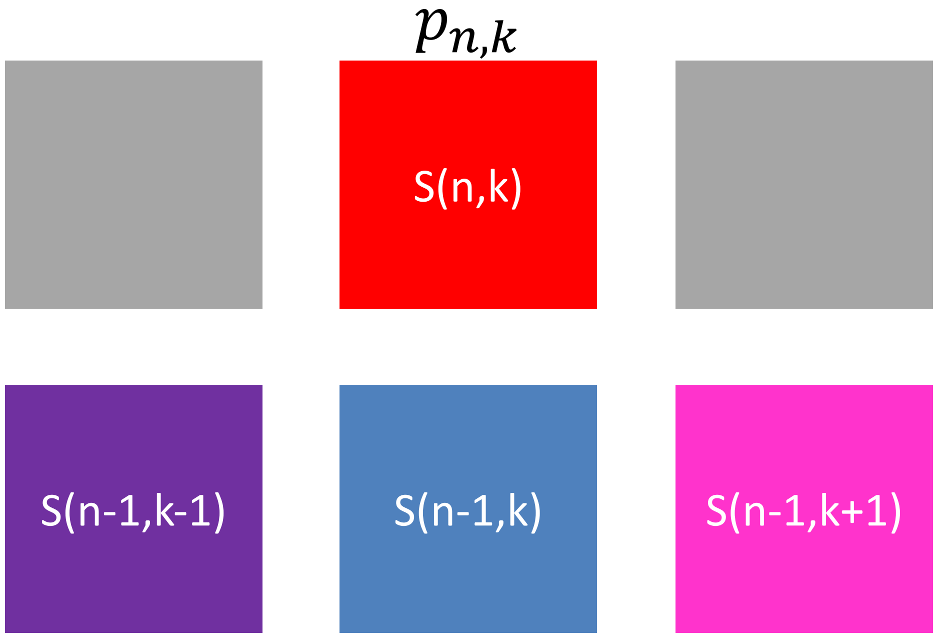
S(n-1,k)

S(n-1,k+1)

Computing $S(n,k)$

Assume we know the least energy seams for all of row $n-1$ (i.e. we know $S(n-1,\ell)$ for all ℓ)

$$S(n,k) = min\begin{cases} S(n-1,k-1)+e(p_{n,k}) \\ S(n-1,k)+e(p_{n,k}) \\ S(n-1,k+1)+e(p_{n,k}) \end{cases}$$



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PA4!

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PA4!

Longest Common Subsequence

Given two sequences X and Y, find the length of their longest common subsequence

Example:

$X = ATCTGAT$

$Y = TGCATA$

$LCS = TCTA$

Brute force: Compare every subsequence of X with Y

$\Omega(2^n)$

Dynamic Programming

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1. Identify Recursive Structure

Let $LCS(i, j)$ = length of the LCS for the first i characters of X, first j character of Y

Find $LCS(i, j)$:

Case 1: $X[i] = Y[j]$

$$X = ATCTGCGT$$
$$Y = TGCATAT$$
$$LCS(i, j) = LCS(i-1, j-1) + 1$$

Case 2: $X[i] \neq Y[j]$

$$X=ATCTGCGA$$
$$Y=TGCATAT$$
$$LCS(i, j) = LCS(i, j-1)$$

$$X=ATCTGCGT$$
$$Y=TGCATAC$$
$$LCS(i, j) = LCS(i-1, j)$$

$$LCS(i, j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

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Read from M[i,j] if present

Save to M[i,j]

```
X = "alkjdflaksjdf"
Y = "lakjsdflkasjdlfs"
M = 2d array of len(X) rows and len(Y) columns, initialized to -1
def LCS(int i, int j):
        # returns the length of the LCS shared between the length-i prefix of  X and length-j prefix of Y
        # memoization
        if M[i,j] > -1:
                return M[i,j]
        #base case:
        if i == 0 or j == 0:
                ans = 0
        elif X[i] == Y[j]:
                ans = LCS(i-1, j-1) + 1
        else:
                ans = max( LCS(i, j-1), LCS(i-1, j) )
        M[i,j] = ans
        return ans
print(LCS(len(X)+1, len(Y)+1)) # the answer for the entirety of X and Y
```

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

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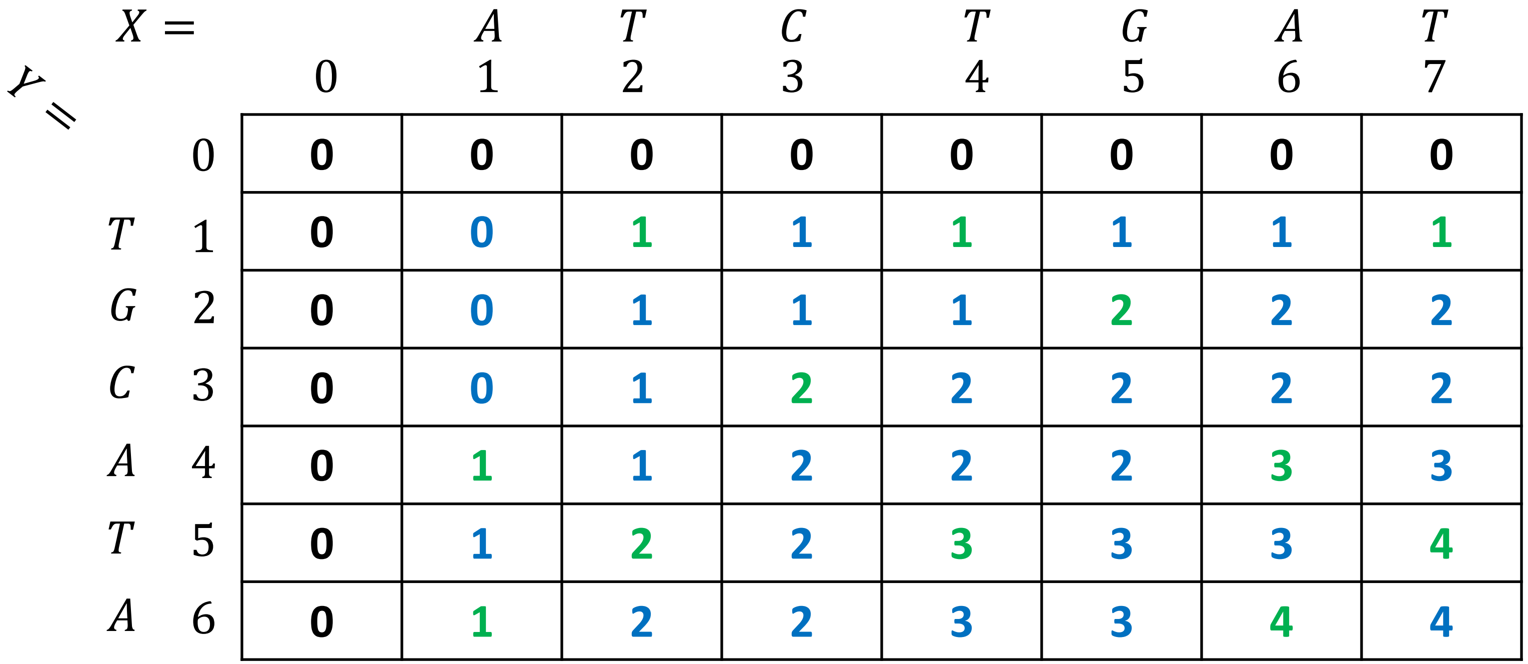
3. Solve in a Good Order

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$



$Y =$	$X =$	0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
	0	0	0	0	0	0	0	0	0
T	1	0	0	1	1	1	1	1	1
G	2	0	0	1	1	1	2	2	2
C	3	0	0	1	2	2	2	2	2
A	4	0	1	1	2	2	2	3	3
T	5	0	1	2	2	3	3	3	4
A	6	0	1	2	2	3	3	4	4

To fill in cell (i, j) we need cells $(i-1, j-1), (i-1, j), (i, j-1)$

Fill from Top->Bottom, Left->Right (with any preference)

Run Time?

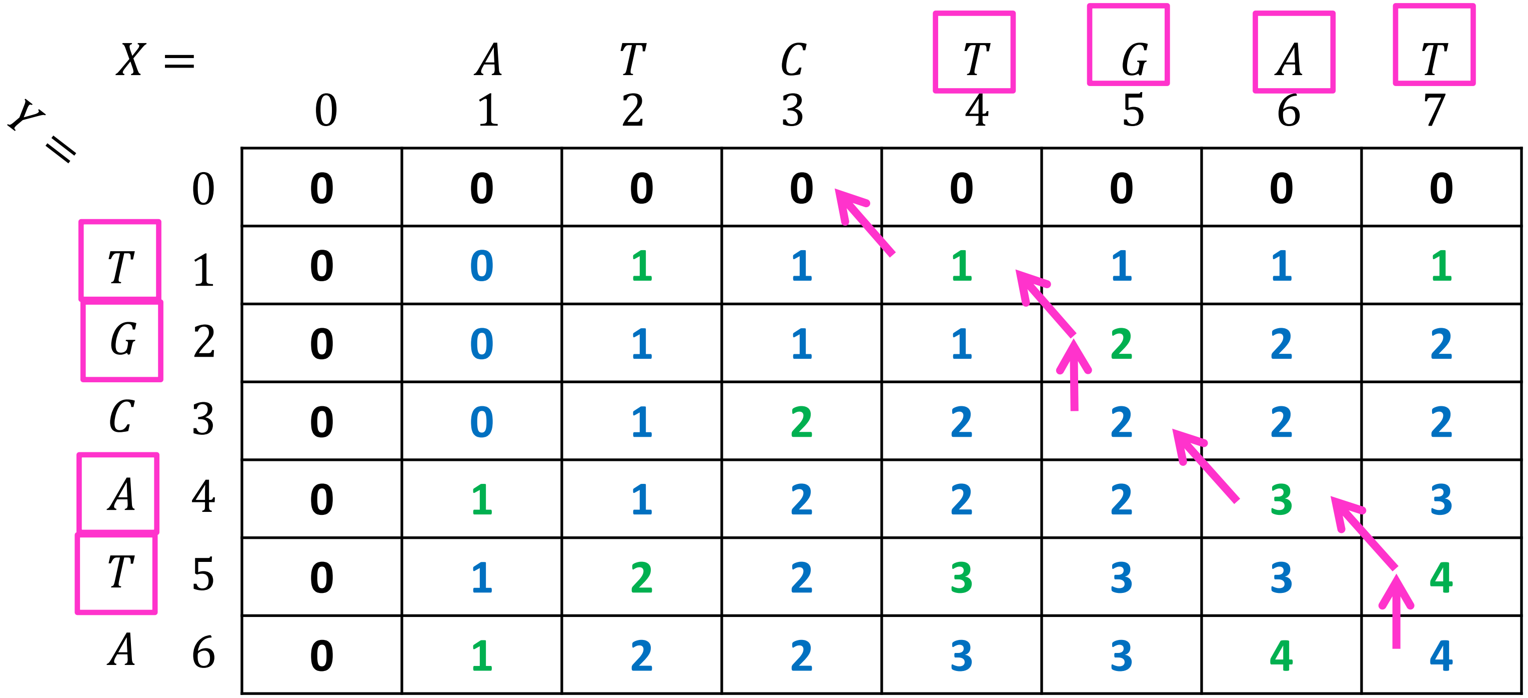
$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$

$Y =$	$X =$	0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
T	1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
G	2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
C	3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
A	4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
T	5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
A	6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

Run Time: $\Theta(n \cdot m)$ (for $|X| = n$, $|Y| = m$)

Reconstructing the LCS

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$



$Y =$	$X =$	0	A 1	T 2	C 3	T 4	G 5	A 6	T 7
	0	0	0	0	0	0	0	0	0
T	1	0	0	1	1	1	1	1	1
G	2	0	0	1	1	1	2	2	2
C	3	0	0	1	2	2	2	2	2
A	4	0	1	1	2	2	2	3	3
T	5	0	1	2	2	3	3	3	4
A	6	0	1	2	2	3	3	4	4

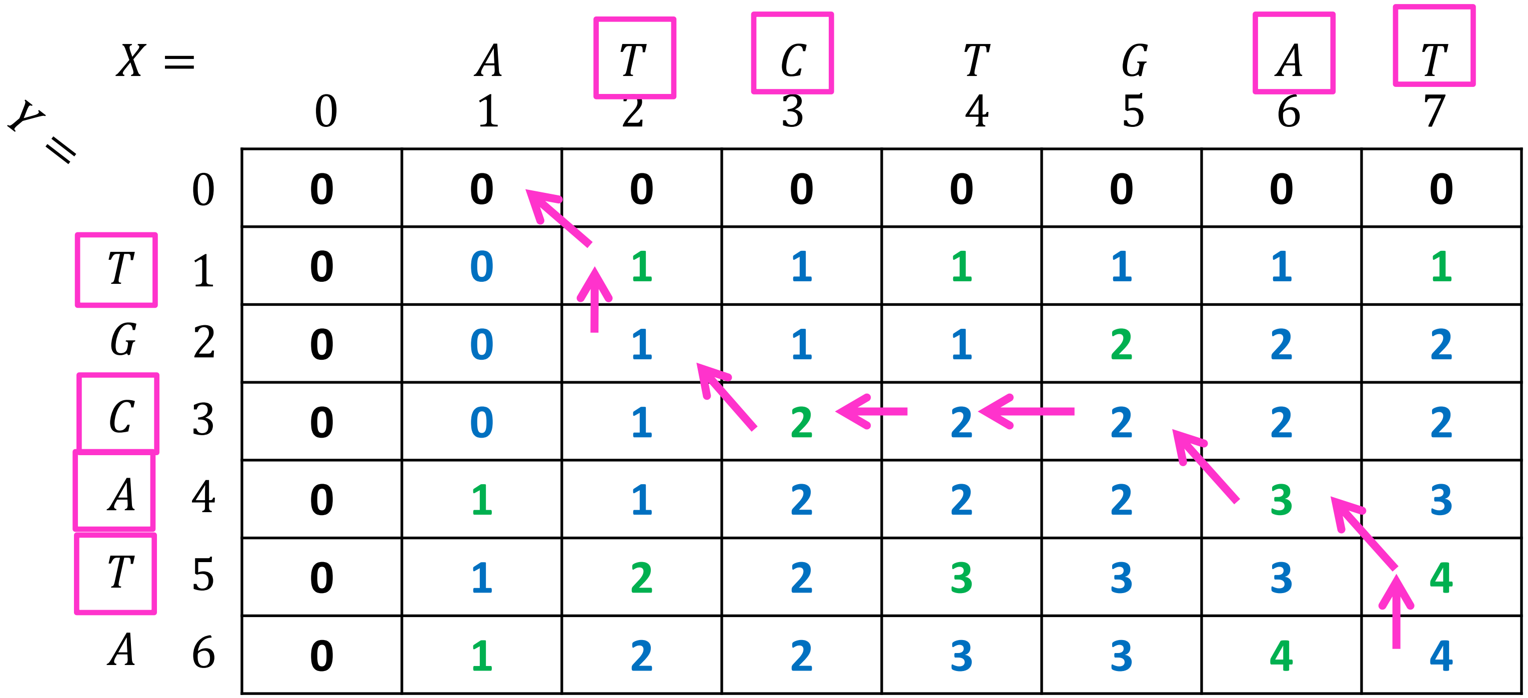
Start from bottom right,

if symbols matched, print that symbol then go diagonally

else go to largest adjacent

Reconstructing the LCS

$$LCS(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS(i-1, j-1) + 1 & \text{if } X[i] = Y[j] \\ \max(LCS(i, j-1), LCS(i-1, j)) & \text{otherwise} \end{cases}$$



	$X =$		A	T	C	T	G	A	T
$Y =$		0	1	2	3	4	5	6	7
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
T	1	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
G	2	**0**	**0**	**1**	**1**	**1**	**2**	**2**	**2**
C	3	**0**	**0**	**1**	**2**	**2**	**2**	**2**	**2**
A	4	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
T	5	**0**	**1**	**2**	**2**	**3**	**3**	**3**	**4**
A	6	**0**	**1**	**2**	**2**	**3**	**3**	**4**	**4**

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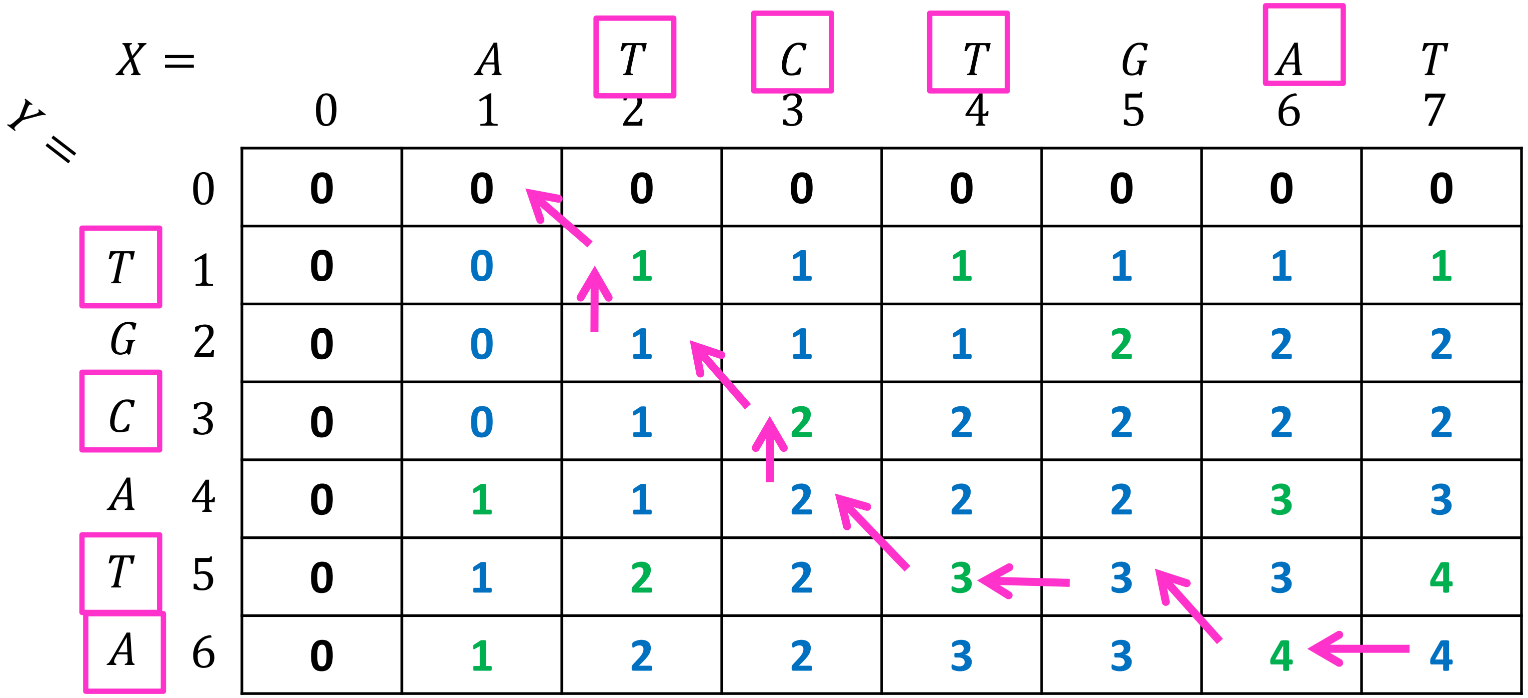

$Y =$	$X =$		A	T	C	T	G	A	T
		0	1	2	3	4	5	6	7
	0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
T	1	**0**	0	1	1	1	1	1	1
G	2	**0**	0	1	1	1	2	2	2
C	3	**0**	0	1	2	2	2	2	2
A	4	**0**	1	1	2	2	2	3	3
T	5	**0**	1	2	2	3	3	3	4
A	6	**0**	1	2	2	3	3	4	4

Start from bottom right,

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